1st SAFEST Olympiad (July 2019), South African - Estonian IMO teams

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– Day 1

1 Let ABC be an isosceles triangle with $AB = AC$. Let AD be the diameter of the circumcircle of ABC and let P be a point on the smaller arc BD. The line DP intersects the rays AB and AC at points M and N, respectively. The line AD intersects the lines BP and CP at points Q and R, respectively. Prove that the midpoint of MN lies on the circumcircle of PQR

2 Define the sequence $a_0, a_1, a_2, \ldots$ by $a_n = 2^n + 2^{\lfloor n/2 \rfloor}$. Prove that there are infinitely many terms of the sequence which can be expressed as a sum of (two or more) distinct terms of the sequence, as well as infinitely many of those which cannot be expressed in such a way.

3 Let $m, n \geq 2$ be integers. Let $f(x_1, \ldots, x_n)$ be a polynomial with real coefficients such that

$$f(x_1, \ldots, x_n) = \left\lfloor \frac{x_1 + \cdots + x_n}{m} \right\rfloor \text{ for every } x_1, \ldots, x_n \in \{0, 1, \ldots, m-1\}.$$

Prove that the total degree of f is at least n.

– Day 2

4 Let $a_1, a_2, ..., a_{2019}$ be any positive real numbers such that $\frac{1}{a_1+2019} + \frac{1}{a_2+2019} + ... + \frac{1}{a_{2019}+2019} = \frac{1}{2019}$.

Find the minimum value of $a_1 a_2 ... a_{2019}$ and determine for which values of $a_1, a_2, ..., a_{2019}$ this minimum occurs

5 There are 25 IMO participants attending a party. Every two of them speak to each other in some language, and they use only one language even if they both know some other language as well. Among every three participants there is a person who uses the same language to speak to the other two (in that group of three). Prove that there is an IMO participant who speaks the same language to at least 10 other participants

6 Let ABC be a triangle with circumcircle Ω and incentre I. A line ℓ intersects the lines AI, BI, and CI at points D, E, and F, respectively, distinct from the points A, B, C, and I. The perpendicular bisectors x, y, and z of the segments AD, BE, and CF, respectively determine a triangle Θ. Show that the circumcircle of the triangle Θ is tangent to Ω.