Online Math Open Problems 2019

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– Spring

1 Daniel chooses some distinct subsets of $\{1, \ldots, 2019\}$ such that any two distinct subsets chosen are disjoint. Compute the maximum possible number of subsets he can choose.

Proposed by Ankan Bhattacharya

2 Let $A = (0, 0), B = (1, 0), C = (-1, 0)$, and $D = (-1, 1)$. Let $\mathcal{C}$ be the closed curve given by the segment AB, the minor arc of the circle $x^2 + (y-1)^2 = 2$ connecting B to C, the segment CD, and the minor arc of the circle $x^2 + (y-1)^2 = 1$ connecting D to A. Let $\mathcal{D}$ be a piece of paper whose boundary is $\mathcal{C}$. Compute the sum of all integers $2 \le n \le 2019$ such that it is possible to cut $\mathcal{D}$ into n congruent pieces of paper.

Proposed by Vincent Huang

3 Compute the smallest positive integer that can be expressed as the product of four distinct integers.

Proposed by Yannick Yao

4 Compute $\left\lceil \sum_{k=2018}^{\infty} \frac{2019! - 2018!}{k!} \right\rceil$. (The notation $\lceil x \rceil$ denotes the least integer n such that $n \ge x$.)

Proposed by Tristan Shin

5 Consider the set S of lattice points (x, y) with $0 \le x, y \le 8$. Call a function $f : S \to \{1, 2, \ldots, 9\}$ a *Sudoku function* if:

- $\{f(x, 0), f(x, 1), \ldots, f(x, 8)\} = \{1, 2, \ldots, 9\}$ for each $0 \le x \le 8$ and $\{f(0, y), f(1, y), \ldots, f(8, y)\} = \{1, 2, \ldots, 9\}$ for each $0 \le y \le 8$.
- For any integers $0 \le m, n \le 2$ and any $0 \le i_1, j_1, i_2, j_2 \le 2$, $f(3m + i_1, 3n + j_1) \ne f(3m + i_2, 3n + j_2)$ unless $i_1 = i_2$ and $j_1 = j_2$.

Over all Sudoku functions f, compute the maximum possible value of $\sum_{0 \le i \le 8} f(i, i) + \sum_{0 \le i \le 7} f(i, i + 1)$.

Proposed by Brandon Wang

6 Let $A, B, C, ..., Z$ be 26 nonzero real numbers. Suppose that $T = TNYWR$. Compute the smallest possible value of

$$\left\lceil A^2 + B^2 + \cdots + Z^2 \right\rceil.$$

(The notation $\lceil x \rceil$ denotes the least integer n such that $n \geq x$.)

Proposed by Luke Robitaille

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7 Let $ABCD$ be a square with side length 4. Consider points P and Q on segments AB and BC, respectively, with $BP = 3$ and $BQ = 1$. Let R be the intersection of AQ and DP. If BR^2 can be expressed in the form $\frac{m}{n}$ for coprime positive integers m, n, compute $m + n$.

Proposed by Brandon Wang

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8 In triangle ABC, side AB has length 10, and the A- and B-medians have length 9 and 12, respectively. Compute the area of the triangle.

Proposed by Yannick Yao

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9 Susan is presented with six boxes $B_1, \dots, B_6$, each of which is initially empty, and two identical coins of denomination 2^k for each $k = 0, \dots, 5$. Compute the number of ways for Susan to place the coins in the boxes such that each box B_k contains coins of total value 2^k.

Proposed by Ankan Bhattacharya

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10 When two distinct digits are randomly chosen in $N = 123456789$ and their places are swapped, one gets a new number N' (for example, if 2 and 4 are swapped, then $N' = 143256789$). The expected value of N' is equal to $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute the remainder when $m + n$ is divided by 10^6.

Proposed by Yannick Yao

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11 Jay is given 99 stacks of blocks, such that the ith stack has i^2 blocks. Jay must choose a positive integer N such that from each stack, he may take either 0 blocks or exactly N blocks. Compute the value Jay should choose for N in order to maximize the number of blocks he may take from the 99 stacks.

Proposed by James Lin

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12 A set D of positive integers is called *indifferent* if there are at least two integers in the set, and for any two distinct elements $x, y \in D$, their positive difference $|x - y|$ is also in D. Let $M(x)$ be the smallest size of an indifferent set whose largest element is x. Compute the sum $M(2) + M(3) + \cdots + M(100)$.

Proposed by Yannick Yao

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13 Let $S = \{10^n + 1000 : n = 0, 1, \dots\}$. Compute the largest positive integer not expressible as the sum of (not necessarily distinct) elements of S.

Proposed by Ankan Bhattacharya

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14 The sum

$$\sum_{i=0}^{1000} \frac{\binom{1000}{i}}{\binom{2019}{i}}$$

can be expressed in the form $\frac{p}{q}$, where p and q are relatively prime positive integers. Compute $p + q$.

Proposed by James Lin

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15 Evan has 66000 omons, particles that can cluster into groups of a perfect square number of omons. An omon in a cluster of n^2 omons has a potential energy of $\frac{1}{n}$. Evan accurately computes the sum of the potential energies of all the omons. Compute the smallest possible value of his result.

Proposed by Michael Ren and Luke Robitaille

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16 In triangle ABC, $BC = 3, CA = 4$, and $AB = 5$. For any point P in the same plane as ABC, define $f(P)$ as the sum of the distances from P to lines AB, BC, and CA. The area of the locus of P where $f(P) \leq 12$ is $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $100m + n$.

Proposed by Yannick Yao

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17 Let $ABCD$ be an isosceles trapezoid with $\overline{AD} \parallel \overline{BC}$. The incircle of $\triangle ABC$ has center I and is tangent to $\overline{BC}$ at P. The incircle of $\triangle ABD$ has center J and is tangent to $\overline{AD}$ at Q. If $PI = 8$, $IJ = 25$, and $JQ = 15$, compute the greatest integer less than or equal to the area of $ABCD$.

Proposed by Ankan Bhattacharya

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18 Define a function f as follows. For any positive integer i, let $f(i)$ be the smallest positive integer j such that there exist pairwise distinct positive integers $a, b, c,$ and d such that $\gcd(a, b)$, $\gcd(a, c)$, $\gcd(a, d)$, $\gcd(b, c)$, $\gcd(b, d)$, and $\gcd(c, d)$ are pairwise distinct and equal to $i, i+1, i+2, i+3, i+4,$ and j in some order, if any such j exists; let $f(i) = 0$ if no such j exists. Compute $f(1) + f(2) + \dots + f(2019)$.

Proposed by Edward Wan

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19 Arianna and Brianna play a game in which they alternate turns writing numbers on a paper. Before the game begins, a referee randomly selects an integer N with $1 \leq N \leq 2019$, such that i has probability $\frac{i}{1+2+\dots+2019}$ of being chosen. First, Arianna writes 1 on the paper. On any move thereafter, the player whose turn it is writes $a+1$ or $2a$, where a is any number on the paper, under the conditions that no number is ever written twice and any number written does not exceed N. No number is ever erased. The winner is the person who first writes the number N. Assuming both Arianna and Brianna play optimally, the probability that Brianna wins can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Compute $m+n$.

Proposed by Edward Wan

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20 Let ABC be a triangle with $AB=4$, $BC=5$, and $CA=6$. Suppose X and Y are points such that

- BC and XY are parallel
- BX and CY intersect at a point P on the circumcircle of $\triangle ABC$
- the circumcircles of $\triangle BCX$ and $\triangle BCY$ are tangent to AB and AC, respectively.

Then AP^2 can be written in the form $\frac{p}{q}$ for relatively prime positive integers p and q. Compute $100p+q$.

Proposed by Tristan Shin

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21 Define a sequence by $a_0=2019$ and $a_n=a_{n-1}^{2019}$ for all positive integers n. Compute the remainder when

$$a_0+a_1+a_2+\cdots+a_{51}$$

is divided by 856.

Proposed by Tristan Shin

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22 For any set S of integers, let $f(S)$ denote the number of integers k with $0\le k<2019$ such that there exist $s_1,s_2\in S$ satisfying $s_1-s_2=k$. For any positive integer m, let x_m be the minimum possible value of $f(S_1)+\cdots+f(S_m)$ where $S_1,\ldots,S_m$ are nonempty sets partitioning the positive integers. Let M be the minimum of $x_1,x_2,\ldots$, and let N be the number of positive integers m such that $x_m=M$. Compute $100M+N$.

Proposed by Ankan Bhattacharya

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23 Let a_1, a_2, a_3, a_4, and a_5 be real numbers satisfying

$$a_1a_2+a_2a_3+a_3a_4+a_4a_5+a_5a_1=20,$$
$$a_1a_3+a_2a_4+a_3a_5+a_4a_1+a_5a_2=22.$$

Then the smallest possible value of $a_1^2+a_2^2+a_3^2+a_4^2+a_5^2$ can be expressed as $m+\sqrt{n}$, where m and n are positive integers. Compute $100m+n$.

Proposed by Ankan Bhattacharya

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24 We define the binary operation $\times$ on elements of $\mathbb{Z}^2$ as

$$(a,b)\times(c,d)=(ac+bd,ad+bc)$$

for all integers a, b, c, and d. Compute the number of ordered six-tuples $(a_1, a_2, a_3, a_4, a_5, a_6)$ of integers such that

$$[[[[(1,a_1)\times(2,a_2)]\times(3,a_3)]\times(4,a_4)]\times(5,a_5)]\times(6,a_6)=(350,280).$$

Proposed by Michael Ren and James Lin

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25 Let S be the set of positive integers not divisible by p^4 for all primes p. Anastasia and Bananastasia play a game.

At the beginning, Anastasia writes down the positive integer N on the board. Then the players take moves in turn; Bananastasia moves first. On any move of his, Bananastasia replaces the number n on the blackboard with a number of the form $n-a$, where $a\in S$ is a positive integer. On any move of hers, Anastasia replaces the number n on the blackboard with a number of the form n^k, where k is a positive integer. Bananastasia wins if the number on the board becomes zero.

Compute the second-smallest possible value of N for which Anastasia can prevent Bananastasia from winning.

Proposed by Brandon Wang and Vincent Huang

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26 There exists a unique prime $p>5$ for which the decimal expansion of $\frac{1}{p}$ repeats with a period of exactly 294. Given that $p>10^{50}$, compute the remainder when p is divided by 10^9.

Proposed by Ankan Bhattacharya

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27 Let G be a graph on n vertices $V_1, V_2, \dots, V_n$ and let $P_1, P_2, \dots, P_n$ be points in the plane. Suppose that, whenever V_i and V_j are connected by an edge, P_iP_j has length 1; in this situation, we say that the P_i form an *embedding* of G in the plane. Consider a set $S\subseteq\{1,2,\dots,n\}$ and a configuration of points Q_i for each $i\in S$. If the number of embeddings of G such that $P_i=Q_i$ for each $i\in S$ is finite and nonzero, we say that S is a *tasty* set. Out of all tasty sets S, we define a function $f(G)$ to be the smallest size of a tasty set. Let T be the set of all connected graphs on n vertices with $n-1$ edges. Choosing G uniformly and at random from T, let a_n be the expected value of $\frac{f(G)^2}{n^2}$. Compute $\left\lfloor 2019\lim_{n\to\infty}a_n\right\rfloor$.

Proposed by Vincent Huang

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28 Let ABC be a triangle. There exists a positive real number x such that $AB=6x^2+1$ and $AC=2x^2+2x$, and there exist points W and X on segment AB along with points Y and Z

on segment AC such that $AW = x$, $WX = x+4$, $AY = x+1$, and $YZ = x$. For any line ℓ not intersecting segment BC, let $f(\ell)$ be the unique point P on line ℓ and on the same side of BC as A such that ℓ is tangent to the circumcircle of triangle PBC. Suppose lines $f(WY)f(XY)$ and $f(WZ)f(XZ)$ meet at B, and that lines $f(WZ)f(WY)$ and $f(XY)f(XZ)$ meet at C. Then the product of all possible values for the length of BC can be expressed in the form $a + \frac{b\sqrt{c}}{d}$ for positive integers a, b, c, d with c squarefree and $\gcd(b, d) = 1$. Compute $100a + b + c + d$.

Proposed by Vincent Huang

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29 Let n be a positive integer and let $P(x)$ be a monic polynomial of degree n with real coefficients. Also let $Q(x) = (x+1)^2(x+2)^2 \dots (x+n+1)^2$. Consider the minimum possible value m_n of $\sum_{i=1}^{n+1} \frac{i^2 P(i^2)^2}{Q(i)}$. Then there exist positive constants a, b, c such that, as n approaches infinity, the ratio between m_n and $a^{2n} n^{2n+b} c$ approaches 1. Compute $\lfloor 2019abc^2 \rfloor$.

Proposed by Vincent Huang

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30 Let ABC be a triangle with symmedian point K, and let $\theta = \angle AKB - 90^\circ$. Suppose that θ is both positive and less than $\angle C$. Consider a point K' inside $\triangle ABC$ such that A, K', K, and B are concyclic and $\angle K'CB = \theta$. Consider another point P inside $\triangle ABC$ such that $K'P \perp BC$ and $\angle PCA = \theta$. If $\sin \angle APB = \sin^2(C - \theta)$ and the product of the lengths of the A- and B-medians of $\triangle ABC$ is $\sqrt{\sqrt{5}+1}$, then the maximum possible value of $5AB^2 - CA^2 - CB^2$ can be expressed in the form $m\sqrt{n}$ for positive integers m, n with n squarefree. Compute $100m + n$.

Proposed by Vincent Huang

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1 Compute the sum of all positive integers n such that the median of the n smallest prime numbers is n.

Proposed by Luke Robitaille

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2 Let A, B, C, and P be points in the plane such that no three of them are collinear. Suppose that the areas of triangles BPC, CPA, and APB are 13, 14, and 15, respectively. Compute the sum of all possible values for the area of triangle ABC.

Proposed by Ankan Bhattacharya

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3 Let k be a positive real number. Suppose that the set of real numbers x such that $x^2 + k|x| \leq 2019$ is an interval of length 6. Compute k.

Proposed by Luke Robitaille

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4 Maryssa, Stephen, and Cynthia played a game. Each of them independently privately chose one of Rock, Paper, and Scissors at random, with all three choices being equally likely. Given that at least one of them chose Rock and at most one of them chose Paper, the probability that exactly one of them chose Scissors can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $100m + n$.

Proposed by Yannick Yao

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5 Compute the number of ordered pairs (m, n) of positive integers that satisfy the equation $\text{lcm}(m, n) + \gcd(m, n) = m + n + 30$.

Proposed by Ankit Bisain

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6 An ant starts at the origin of the Cartesian coordinate plane. Each minute it moves randomly one unit in one of the directions up, down, left, or right, with all four directions being equally likely; its direction each minute is independent of its direction in any previous minutes. It stops when it reaches a point (x, y) such that $|x| + |y| = 3$. The expected number of moves it makes before stopping can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $100m + n$.

Proposed by Yannick Yao

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7 At a concert 10 singers will perform. For each singer x, either there is a singer y such that x wishes to perform right after y, or x has no preferences at all. Suppose that there are n ways to order the singers such that no singer has an unsatisfied preference, and let p be the product of all possible nonzero values of n. Compute the largest nonnegative integer k such that 2^k divides p.

Proposed by Gopal Goel

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8 There are three eight-digit positive integers which are equal to the sum of the eighth powers of their digits. Given that two of the numbers are 24678051 and 88593477, compute the third number.

Proposed by Vincent Huang

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9 Convex equiangular hexagon $ABCDEF$ has $AB = CD = EF = 1$ and $BC = DE = FA = 4$. Congruent and pairwise externally tangent circles γ_1, γ_2, and γ_3 are drawn such that γ_1 is tangent to side $\overline{AB}$ and side $\overline{BC}$, γ_2 is tangent to side $\overline{CD}$ and side $\overline{DE}$, and γ_3 is tangent to side $\overline{EF}$ and side $\overline{FA}$. Then the area of γ_1 can be expressed as $\frac{m\pi}{n}$ for relatively prime positive integers m and n. Compute $100m + n$.

Proposed by Sean Li

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10 Let k be a positive integer. Marco and Vera play a game on an infinite grid of square cells. At the beginning, only one cell is black and the rest are white.

A turn in this game consists of the following. Marco moves first, and for every move he must choose a cell which is black and which has more than two white neighbors. (Two cells are neighbors if they share an edge, so every cell has exactly four neighbors.) His move consists of making the chosen black cell white and turning all of its neighbors black if they are not already. Vera then performs the following action exactly k times: she chooses two cells that are neighbors to each other and swaps their colors (she is allowed to swap the colors of two white or of two black cells, though doing so has no effect). This, in totality, is a single turn. If Vera leaves the board so that Marco cannot choose a cell that is black and has more than two white neighbors, then Vera wins; otherwise, another turn occurs.
Let m be the minimal k value such that Vera can guarantee that she wins no matter what Marco does. For $k = m$, let t be the smallest positive integer such that Vera can guarantee, no matter what Marco does, that she wins after at most t turns. Compute $100m + t$.

Proposed by Ashwin Sah

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11 Let ABC be a triangle with incenter I such that $AB = 20$ and $AC = 19$. Point $P \neq A$ lies on line AB and point $Q \neq A$ lies on line AC. Suppose that $IA = IP = IQ$ and that line PQ passes through the midpoint of side BC. Suppose that $BC = \frac{m}{n}$ for relatively prime positive integers m and n. Compute $100m + n$.

Proposed by Ankit Bisain

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12 Let $F(n)$ denote the smallest positive integer greater than n whose sum of digits is equal to the sum of the digits of n. For example, $F(2019) = 2028$. Compute $F(1) + F(2) + \cdots + F(1000)$.

Proposed by Sean Li

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13 Compute the number of subsets S with at least two elements of $\{2^2, 3^3, \ldots, 216^{216}\}$ such that the product of the elements of S has exactly 216 positive divisors.

Proposed by Sean Li

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14 The sequence of nonnegative integers $F_0, F_1, F_2, \ldots$ is defined recursively as $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for all integers $n \geq 0$. Let d be the largest positive integer such that, for all integers $n \geq 0$, d divides $F_{n+2020} - F_n$. Compute the remainder when d is divided by 1001.

Proposed by Ankit Bisain

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15 Let A,B,C, and D be points in the plane with $AB = AC = BC = BD = CD = 36$ and such that $A \neq D$. Point K lies on segment AC such that $AK = 2KC$. Point M lies on segment AB, and point N lies on line AC, such that D, M, and N are collinear. Let lines CM and BN intersect at P. Then the maximum possible length of segment KP can be expressed in the form $m + \sqrt{n}$ for positive integers m and n. Compute $100m + n$.

Proposed by James Lin

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16 Let ABC be a scalene triangle with inradius 1 and exradii r_A, r_B, and r_C such that

$$20\left(r_B^2r_C^2+r_C^2r_A^2+r_A^2r_B^2\right)=19\left(r_Ar_Br_C\right)^2.$$

If

$$\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}=2.019,$$

then the area of $\triangle ABC$ can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $100m+n$.

Proposed by Tristan Shin

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17 For an ordered pair (m,n) of distinct positive integers, suppose, for some nonempty subset S of $\mathbb{R}$, that a function $f:S\rightarrow S$ satisfies the property that $f^m(x)+f^n(y)=x+y$ for all $x,y\in S$. (Here $f^k(z)$ means the result when f is applied k times to z; for example, $f^1(z)=f(z)$ and $f^3(z)=f(f(f(z)))$.) Then f is called (m,n)-*splendid*. Furthermore, f is called (m,n)-*primitive* if f is (m,n)-splendid and there do not exist positive integers $a\le m$ and $b\le n$ with $(a,b)\neq(m,n)$ and $a\neq b$ such that f is also (a,b)-splendid. Compute the number of ordered pairs (m,n) of distinct positive integers less than 10000 such that there exists a nonempty subset S of $\mathbb{R}$ such that there exists an (m,n)-primitive function $f:S\rightarrow S$.

Proposed by Vincent Huang

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18 Define a *modern artwork* to be a nonempty finite set of rectangles in the Cartesian coordinate plane with positive areas, pairwise disjoint interiors, and sides parallel to the coordinate axes. For a modern artwork S, define its *price* to be the minimum number of colors with which Sean could paint the interiors of rectangles in S such that every rectangle's interior is painted in exactly one color and every two distinct touching rectangles have distinct colors, where two rectangles are *touching* if they share infinitely many points. For a positive integer n, let $g(n)$ denote the maximum price of any modern artwork with exactly n rectangles. Compute $g(1)+g(2)+\cdots+g(2019)$.

Proposed by Yang Liu and Edward Wan

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19 Let ABC be an acute triangle with circumcenter O and orthocenter H. Let E be the intersection of BH and AC and let M and N be the midpoints of HB and HO, respectively. Let I be the incenter of AEM and J be the intersection of ME and AI. If $AO=20$, $AN=17$, and $\angle ANM=90^\circ$, then $\frac{AI}{AJ}=\frac{m}{n}$ for relatively prime positive integers m and n. Compute $100m+n$.

Proposed by Tristan Shin

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20 Define a *crossword puzzle* to be a 15×15 grid of squares, each of which is either black or white. In a crossword puzzle, define a *word* to be a sequence of one or more consecutive white squares in a row or column such that the squares immediately before and after the sequence both are either black or nonexistent. (The latter case would occur if an end of a word coincides with an end of a row or column of the grid.) A crossword puzzle is *tasty* if every word consists of an even number of white squares. Compute the sum of all nonnegative integers n such that there exists a tasty crossword puzzle with exactly n white squares.

Proposed by Luke Robitaille

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21 Let p and q be prime numbers such that $(p-1)^{q-1}-1$ is a positive integer that divides $(2q)^{2p}-1$. Compute the sum of all possible values of pq.

Proposed by Ankit Bisain

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22 For finite sets A and B, call a function $f : A \to B$ an *antibijection* if there does not exist a set $S \subseteq A \cap B$ such that S has at least two elements and, for all $s \in S$, there exists exactly one element s' of S such that $f(s') = s$. Let N be the number of antibijections from $\{1, 2, 3, \dots 2018\}$ to $\{1, 2, 3, \dots 2019\}$. Suppose N is written as the product of a collection of (not necessarily distinct) prime numbers. Compute the sum of the members of this collection. (For example, if it were true that $N = 12 = 2 \times 2 \times 3$, then the answer would be $2 + 2 + 3 = 7$.)

Proposed by Ankit Bisain

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23 Let v and w be real numbers such that, for all real numbers a and b, the inequality

$$(2^{a+b} + 8)(3^a + 3^b) \leq v(12^{a-1} + 12^{b-1} - 2^{a+b-1}) + w$$

holds. Compute the smallest possible value of $128v^2 + w^2$.

Proposed by Luke Robitaille

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24 Let ABC be an acute scalene triangle with orthocenter H and circumcenter O. Let the line through A tangent to the circumcircle of triangle AHO intersect the circumcircle of triangle ABC at A and $P \neq A$. Let the circumcircles of triangles AOP and BHP intersect at P and $Q \neq P$. Let line PQ intersect segment BO at X. Suppose that $BX = 2$, $OX = 1$, and $BC = 5$. Then $AB \cdot AC = \sqrt{k} + m\sqrt{n}$ for positive integers k, m, and n, where neither k nor n is divisible by the square of any integer greater than 1. Compute $100k + 10m + n$.

Proposed by Luke Robitaille

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25 The sequence $f_0, f_1, \dots$ of polynomials in $\mathbb{F}_{11}[x]$ is defined by $f_0(x) = x$ and $f_{n+1}(x) = f_n(x)^{11} - f_n(x)$ for all $n \geq 0$. Compute the remainder when the number of nonconstant monic irreducible divisors of $f_{1000}(x)$ is divided by 1000.

Proposed by Ankan Bhattacharya

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26 Let $p = 491$ be prime. Let S be the set of ordered k-tuples of nonnegative integers that are less than p. We say that a function $f\colon S \to S$ is k-*murine* if, for all $u, v \in S$, $\langle f(u), f(v)\rangle \equiv \langle u, v\rangle \pmod p$, where $\langle (a_1, \dots, a_k), (b_1, \dots, b_k)\rangle = a_1b_1 + \cdots + a_kb_k$ for any $(a_1, \dots a_k), (b_1, \dots b_k) \in S$.

Let $m(k)$ be the number of k-murine functions. Compute the remainder when $m(1)+m(2)+m(3)+\cdots+m(p)$ is divided by 488.

Proposed by Brandon Wang

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27 A *complex set*, along with its *complexity*, is defined recursively as the following:

-The set $\mathbb{C}$ of complex numbers is a complex set with complexity 1.
-Given two complex sets C_1, C_2 with complexity c_1, c_2 respectively, the set of all functions $f : C_1 \to C_2$ is a complex set denoted $[C_1, C_2]$ with complexity $c_1 + c_2$.

A *complex expression*, along with its *evaluation* and its *complexity*, is defined recursively as the following:

-A single complex set C with complexity c is a complex expression with complexity c that evaluates to itself.
-Given two complex expressions E_1, E_2 with complexity e_1, e_2 that evaluate to C_1 and C_2 respectively, if $C_1 = [C_2, C]$ for some complex set C, then (E_1, E_2) is a complex expression with complexity $e_1 + e_2$ that evaluates to C.

For a positive integer n, let a_n be the number of complex expressions with complexity n that evaluate to $\mathbb{C}$. Let x be a positive real number. Suppose that

$$a_1 + a_2x + a_3x^2 + \cdots = \frac{7}{4}.$$

Then $x = \frac{k\sqrt{m}}{n}$, where k,m, and n are positive integers such that m is not divisible by the square of any integer greater than 1, and k and n are relatively prime. Compute $100k + 10m + n$.

Proposed by Luke Robitaille and Yannick Yao

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28 Let S be the set of integers modulo 2020. Suppose that $a_1, a_2, ..., a_{2020}, b_1, b_2, ..., b_{2020}, c$ are arbitrary elements of S. For any $x_1, x_2, ..., x_{2020} \in S$, define $f(x_1, x_2, ..., x_{2020})$ to be the 2020-tuple whose ith coordinate is $x_{i-2} + a_ix_{2019} + b_ix_{2020} + cx_i$, where we set $x_{-1} = x_0 = 0$. Let m be the smallest positive integer such that, for some values of $a_1, a_2, ..., a_{2020}, b_1, b_2, ..., b_{2020}, c$, we have, for all $x_1, x_2, ..., x_{2020} \in S$, that $f^m(x_1, x_2, ..., x_{2020}) = (0, 0, ..., 0)$. For this value of m, there are exactly n choices of the tuple $(a_1, a_2, ..., a_{2020}, b_1, b_2, ..., b_{2020}, c)$ such that, for all $x_1, x_2, ..., x_{2020} \in S$, $f^m(x_1, x_2, ..., x_{2020}) = (0, 0, ..., 0)$. Compute $100m + n$.

Proposed by Vincent Huang

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29 Let ABC be a triangle. The line through A tangent to the circumcircle of ABC intersects line BC at point W. Points $X, Y \neq A$ lie on lines AC and AB, respectively, such that $WA = WX = WY$. Point X_1 lies on line AB such that $\angle AXX_1 = 90^\circ$, and point X_2 lies on line AC such that $\angle AX_1X_2 = 90^\circ$. Point Y_1 lies on line AC such that $\angle AYY_1 = 90^\circ$, and point Y_2 lies on line AB such that $\angle AY_1Y_2 = 90^\circ$. Let lines AW and XY intersect at point Z, and let point P be the foot of the perpendicular from A to line X_2Y_2. Let line ZP intersect line BC at U and the perpendicular bisector of segment BC at V. Suppose that C lies between B and U. Let x be a positive real number. Suppose that $AB = x+1$, $AC = 3$, $AV = x$, and $\frac{BC}{CU} = x$. Then $x = \frac{\sqrt{k}-m}{n}$ for positive integers k,m, and n such that k is not divisible by the square of any integer greater than 1. Compute $100k + 10m + n$.

Proposed by Ankit Bisain, Luke Robitaille, and Brandon Wang

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30 For a positive integer n, we say an n-*transposition* is a bijection $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ such that there exist exactly two elements i of $\{1, 2, \ldots, n\}$ such that $\sigma(i) \neq i$.
Fix some four pairwise distinct n-transpositions $\sigma_1, \sigma_2, \sigma_3, \sigma_4$. Let q be any prime, and let $\mathbb{F}_q$ be the integers modulo q. Consider all functions $f : (\mathbb{F}_q^n)^n \to \mathbb{F}_q$ that satisfy, for all integers i with $1 \leq i \leq n$ and all $x_1, \ldots x_{i-1}, x_{i+1}, \ldots, x_n, y, z \in \mathbb{F}_q^n$,

$$f(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) + f(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n) = f(x_1, \ldots, x_{i-1}, y+z, x_{i+1}, \ldots, x_n),$$

and that satisfy, for all $x_1, \ldots, x_n \in \mathbb{F}_q^n$ and all $\sigma \in \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$,

$$f(x_1, \ldots, x_n) = -f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

(Note that the equalities in the previous sentence are in $\mathbb{F}_q$. Note that, for any $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{F}_q$, we have $(a_1, \ldots, a_n) + (b_1, \ldots, b_n) = (a_1 + b_1, \ldots, a_n + b_n)$, where $a_1 + b_1, \ldots, a_n + b_n \in \mathbb{F}_q$.)
For a given tuple $(x_1, \ldots, x_n) \in (\mathbb{F}_q^n)^n$, let $g(x_1, \ldots, x_n)$ be the number of different values of $f(x_1, \ldots, x_n)$ over all possible functions f satisfying the above conditions.
Pick $(x_1, \ldots, x_n) \in (\mathbb{F}_q^n)^n$ uniformly at random, and let $\varepsilon(q, \sigma_1, \sigma_2, \sigma_3, \sigma_4)$ be the expected value of $g(x_1, \ldots, x_n)$. Finally, let

$$\kappa(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = -\lim_{q \to \infty} \log_q \left(-\ln \left(\frac{\varepsilon(q, \sigma_1, \sigma_2, \sigma_3, \sigma_4) - 1}{q-1} \right) \right).$$

Pick four pairwise distinct n-transpositions $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ uniformly at random from the set of all n-transpositions. Let $\pi(n)$ denote the expected value of $\kappa(\sigma_1, \ldots, \sigma_4)$. Suppose that $p(x)$ and $q(x)$ are polynomials with real coefficients such that $q(-3) \neq 0$ and such that $\pi(n) = \frac{p(n)}{q(n)}$ for infinitely many positive integers n. Compute $\frac{p(-3)}{q(-3)}$.

Proposed by Gopal Goel
