Online Math Open Problems 2018

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– Spring

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1 Farmer James has three types of cows on his farm. A cow with zero legs is called a *ground beef*, a cow with one leg is called a *steak*, and a cow with two legs is called a *lean beef*. Farmer James counts a total of 20 cows and 18 legs on his farm. How many more *ground beef*s than *lean beef*s does Farmer James have?

Proposed by James Lin

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2 The area of a circle (in square inches) is numerically larger than its circumference (in inches). What is the smallest possible integral area of the circle, in square inches?

Proposed by James Lin

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3 Hen Hao randomly selects two distinct squares on a standard 8×8 chessboard. Given that the two squares touch (at either a vertex or a side), the probability that the two squares are the same color can be expressed in the form $\frac{m}{n}$ for relatively prime positive integers m and n. Find $100m + n$.

[i]Proposed by James Lin

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4 Define $f(x) = |x-1|$. Determine the number of real numbers x such that $f(f(\cdots f(f(x))\cdots)) = 0$, where there are 2018 f's in the equation.

[i]Proposed by Yannick Yao

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5 A mouse has a wheel of cheese which is cut into 2018 slices. The mouse also has a 2019-sided die, with faces labeled $0, 1, 2, \ldots, 2018$, and with each face equally likely to come up. Every second, the mouse rolls the dice. If the dice lands on k, and the mouse has at least k slices of cheese remaining, then the mouse eats k slices of cheese; otherwise, the mouse does nothing. What is the expected number of seconds until all the cheese is gone?

Proposed by Brandon Wang

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6 Let $f(x) = x^2 + x$ for all real x. There exist positive integers m and n, and distinct nonzero real numbers y and z, such that $f(y) = f(z) = m + \sqrt{n}$ and $f(\frac{1}{y}) + f(\frac{1}{z}) = \frac{1}{10}$. Compute $100m + n$.

Proposed by Luke Robitaille

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7 A quadrilateral and a pentagon (both not self-intersecting) intersect each other at N distinct points, where N is a positive integer. What is the maximal possible value of N?

[i]Proposed by James Lin

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8 Compute the number of ordered quadruples (a,b,c,d) of distinct positive integers such that $\binom{\binom{a}{b}}{\binom{c}{d}} = 21.$

Proposed by Luke Robitaille

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9 Let k be a positive integer. In the coordinate plane, circle ω has positive integer radius and is tangent to both axes. Suppose that ω passes through $(1, 1000+k)$. Compute the smallest possible value of k.

[i]Proposed by Luke Robitaille

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10 The one hundred U.S. Senators are standing in a line in alphabetical order. Each senator either always tells the truth or always lies. The ith person in line says:

"Of the $101-i$ people who are not ahead of me in line (including myself), more than half of them are truth-tellers."

How many possibilities are there for the set of truth-tellers on the U.S. Senate?

Proposed by James Lin

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11 Lunasa, Merlin, and Lyrica are performing in a concert. Each of them will perform two different solos, and each pair of them will perform a duet, for nine distinct pieces in total. Since the performances are very demanding, no one is allowed to perform in two pieces in a row. In how many different ways can the pieces be arranged in this concert?

Proposed by Yannick Yao

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12 Near the end of a game of Fish, Celia is playing against a team consisting of Alice and Betsy. Each of the three players holds two cards in their hand, and together they have the Nine, Ten, Jack, Queen, King, and Ace of Spades (this set of cards is known by all three players). Besides the two cards she already has, each of them has no information regarding the other two's hands (In particular, teammates Alice and Betsy do not know each other's cards).

It is currently Celia's turn. On a player's turn, the player must ask a player on the other team whether she has a certain card that is in the set of six cards but *not* in the asker's hand. If the player being asked does indeed have the card, then she must reveal the card and put it in the askers hand, and the asker shall ask again (but may ask a different player on the other team); otherwise, she refuses and it is now her turn. Moreover, a card may not be asked if it is known (to the asker) to be not in the asked person's hand. The game ends when all six cards belong to one team, and the team with all the cards wins. Under optimal play, the probability that Celia wins the game is $\frac{p}{q}$ for relatively prime positive integers p and q. Find $100p + q$.

Proposed by Yannick Yao

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13 Find the smallest positive integer n for which the polynomial

$$x^n - x^{n-1} - x^{n-2} - \cdots - x - 1$$

has a real root greater than 1.999.

[i]Proposed by James Lin

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14 Let ABC be a triangle with $AB = 20$ and $AC = 18$. E is on segment AC and F is on segment AB such that $AE = AF = 8$. Let BE and CF intersect at G. Given that $AEGF$ is cyclic, then $BC = m\sqrt{n}$ for positive integers m and n such that n is not divisible by the square of any prime. Compute $100m + n$.

Proposed by James Lin

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15 Let $\mathbb{N}$ denote the set of positive integers. For how many positive integers $k \le 2018$ do there exist a function $f : \mathbb{N} \to \mathbb{N}$ such that $f(f(n)) = 2n$ for all $n \in \mathbb{N}$ and $f(k) = 2018$?

[i]Proposed by James Lin

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16 In a rectangular 57×57 grid of cells, k of the cells are colored black. What is the smallest positive integer k such that there must exist a rectangle, with sides parallel to the edges of the grid, that has its four vertices at the center of distinct black cells?

[i]Proposed by James Lin

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17 Let S be the set of all subsets of $\{2, 3, \ldots, 2016\}$ with size 1007, and for a nonempty set T of numbers, let $f(T)$ be the product of the elements in T. Determine the remainder when

$$\sum_{T \in S} \left(f(T) - f(T)^{-1}\right)^2$$

is divided by 2017. Note: For b relatively prime to 2017, we say that b^{-1} is the unique positive integer less than 2017 for which 2017 divides $bb^{-1} - 1$.

Proposed by Tristan Shin

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18 Suppose that a, b, c are real numbers such that $a < b < c$ and $a^3 - 3a + 1 = b^3 - 3b + 1 = c^3 - 3c + 1 = 0$. Then $\frac{1}{a^2+b} + \frac{1}{b^2+c} + \frac{1}{c^2+a}$ can be written as $\frac{p}{q}$ for relatively prime positive integers p and q. Find $100p + q$.

Proposed by Michael Ren

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19 Let $P(x)$ be a polynomial of degree at most 2018 such that $P(i) = \binom{2018}{i}$ for all integer i such that $0 \le i \le 2018$. Find the largest nonnegative integer n such that $2^n \mid P(2020)$.

[i]Proposed by Michael Ren

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20 Let ABC be a triangle with $AB = 7, BC = 5$, and $CA = 6$. Let D be a variable point on segment BC, and let the perpendicular bisector of AD meet segments AC, AB at E, F, respectively. It is given that there is a point P inside $\triangle ABC$ such that $\frac{AP}{PC} = \frac{AE}{EC}$ and $\frac{AP}{PB} = \frac{AF}{FB}$. The length of the path traced by P as D varies along segment BC can be expressed as $\sqrt{\frac{m}{n}} \sin^{-1}\left(\sqrt{\frac{1}{7}}\right)$, where m and n are relatively prime positive integers, and angles are measured in radians. Compute $100m + n$.

Proposed by Edward Wan

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21 Let $\bigoplus$ and $\bigotimes$ be two binary boolean operators, i.e. functions that send $\{\text{True}, \text{False}\} \times \{\text{True}, \text{False}\}$ to $\{\text{True}, \text{False}\}$. Find the number of such pairs $(\bigoplus, \bigotimes)$ such that $\bigoplus$ and $\bigotimes$ distribute over each other, that is, for any three boolean values a, b, c, the following four equations hold:

1) $c \bigotimes (a \bigoplus b) = (c \bigotimes a) \bigoplus (c \bigotimes b)$;
2) $(a \bigoplus b) \bigotimes c = (a \bigotimes c) \bigoplus (b \bigotimes c)$;
3) $c \bigoplus (a \bigotimes b) = (c \bigoplus a) \bigotimes (c \bigoplus b)$;
4) $(a \bigotimes b) \bigoplus c = (a \bigoplus c) \bigotimes (b \bigoplus c)$.

[i]Proposed by Yannick Yao

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22 Let $p = 9001$ be a prime number and let $\mathbb{Z}/p\mathbb{Z}$ denote the additive group of integers modulo p. Furthermore, if $A, B \subset \mathbb{Z}/p\mathbb{Z}$, then denote $A + B = \{a + b \pmod{p} | a \in A, b \in B\}$. Let $s_1, s_2, \dots, s_8$ are positive integers that are at least 2. Yang the Sheep notices that no matter how he chooses sets $T_1, T_2, \dots, T_8 \subset \mathbb{Z}/p\mathbb{Z}$ such that $|T_i| = s_i$ for $1 \le i \le 8, T_1 + T_2 + \dots + T_7$ is never equal to $\mathbb{Z}/p\mathbb{Z}$, but $T_1 + T_2 + \dots + T_8$ must always be exactly $\mathbb{Z}/p\mathbb{Z}$. What is the minimum possible value of s_8?

[i]Proposed by Yang Liu

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23 Let ABC be a triangle with $BC = 13, CA = 11, AB = 10$. Let A_1 be the midpoint of BC. A variable line ℓ passes through A_1 and meets AC, AB at B_1, C_1. Let B_2, C_2 be points with $B_2B = B_2C, B_2C_1 \perp AB, C_2B = C_2C, C_2B_1 \perp AC$, and define $P = BB_2 \cap CC_2$. Suppose the circles of diameters BB_2, CC_2 meet at a point $Q \ne A_1$. Given that Q lies on the same side of line BC as A, the minimum possible value of $\frac{PB}{PC} + \frac{QB}{QC}$ can be expressed in the form $\frac{a\sqrt{b}}{c}$ for positive integers a, b, c with $\gcd(a, c) = 1$ and b squarefree. Determine $a + b + c$.

Proposed by Vincent Huang

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24 Find the number of ordered triples (a, b, c) of integers satisfying $0 \le a, b, c \le 1000$ for which

$$a^3 + b^3 + c^3 \equiv 3abc + 1 \pmod{1001}.$$

Proposed by James Lin

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25 Let m and n be positive integers. Fuming Zeng gives James a rectangle, such that $m-1$ lines are drawn parallel to one pair of sides and $n-1$ lines are drawn parallel to the other pair of sides (with each line distinct and intersecting the interior of the rectangle), thus dividing the rectangle into an $m \times n$ grid of smaller rectangles. Fuming Zeng chooses $m+n-1$ of the mn smaller rectangles and then tells James the area of each of the smaller rectangles. Of the $\binom{mn}{m+n-1}$ possible combinations of rectangles and their areas Fuming Zeng could have given, let $C_{m,n}$ be the number of combinations which would allow James to determine the area of the whole rectangle. Given that

$$A = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{C_{m,n}\binom{m+n}{m}}{(m+n)^{m+n}},$$

then find the greatest integer less than $1000A$.

[i]Proposed by James Lin

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26 Let ABC be a triangle with incenter I. Let P and Q be points such that $IP \perp AC$, $IQ \perp AB$, and $IA \perp PQ$. Assume that BP and CQ intersect at the point $R \neq A$ on the circumcircle of ABC such that $AR \parallel BC$. Given that $\angle B - \angle C = 36^\circ$, the value of $\cos A$ can be expressed in the form $\frac{m-\sqrt{n}}{p}$ for positive integers m, n, p and where n is not divisible by the square of any prime. Find the value of $100m + 10n + p$.

Proposed by Michael Ren

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27 Let $n = 2^{2018}$ and let $S = \{1, 2, \ldots, n\}$. For subsets $S_1, S_2, \ldots, S_n \subseteq S$, we call an ordered pair (i, j) *murine* if and only if $\{i, j\}$ is a subset of at least one of S_i, S_j. Then, a sequence of subsets $(S_1, \ldots, S_n)$ of S is called *tasty* if and only if:

1) For all i, $i \in S_i$.
2) For all i, $\bigcup_{j \in S_i} S_j = S_i$.
3) There do not exist pairwise distinct integers $a_1, a_2, \ldots, a_k$ with $k \ge 3$ such that for each i, (a_i, a_{i+1}) is murine, where indices are taken modulo k.
4) n divides $1 + |S_1| + |S_2| + \ldots + |S_n|$.

Find the largest integer x such that 2^x divides the number of tasty sequences $(S_1, \ldots, S_n)$.

[i]Proposed by Vincent Huang and Brandon Wang

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28 In $\triangle ABC$, the incircle ω has center I and is tangent to $\overline{CA}$ and $\overline{AB}$ at E and F respectively. The circumcircle of $\triangle BIC$ meets ω at P and Q. Lines AI and BC meet at D, and the circumcircle of $\triangle PDQ$ meets $\overline{BC}$ again at X. Suppose that $EF = PQ = 16$ and $PX + QX = 17$. Then BC^2 can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

[i]Proposed by Ankan Bhattacharya and Michael Ren

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29 Let $q < 50$ be a prime number. Call a sequence of polynomials $P_0(x), P_1(x), P_2(x), ..., P_{q^2}(x)$ *tasty* if it satisfies the following conditions:

- P_i has degree i for each i (where we consider constant polynomials, including the 0 polynomial, to have degree 0)
- The coefficients of P_i are integers between 0 and $q-1$ for each i.
- For any $0 \le i, j \le q^2$, the polynomial $P_i(P_j(x)) - P_j(P_i(x))$ has all its coefficients divisible by q.

As q varies over all such prime numbers, determine the total number of tasty sequences of polynomials.

Proposed by Vincent Huang

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30 Let $p = 2017$. Given a positive integer n, an $n \times n$ matrix A is formed with each element a_{ij} randomly selected, with equal probability, from $\{0, 1, \ldots, p-1\}$. Let q_n be probability that $\det A \equiv 1 \pmod{p}$. Let $q = \lim_{n\to\infty} q_n$. If $d_1, d_2, d_3, \ldots$ are the digits after the decimal point in the base p expansion of q, then compute the remainder when $\sum_{k=1}^{p^2} d_k$ is divided by 10^9.

Proposed by Ashwin Sah

– Fall

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1 Leonhard has five cards. Each card has a nonnegative integer written on it, and any two cards show relatively prime numbers. Compute the smallest possible value of the sum of the numbers on Leonhard's cards.

Note: Two integers are relatively prime if no positive integer other than 1 divides both numbers.

[i]Proposed by ABCDE and Tristan Shin

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2 Let $(p_1, p_2, \ldots) = (2, 3, \ldots)$ be the list of all prime numbers, and $(c_1, c_2, \ldots) = (4, 6, \ldots)$ be the list of all composite numbers, both in increasing order. Compute the sum of all positive integers n such that $|p_n - c_n| < 3$.

Proposed by Brandon Wang

3 Katie has a list of real numbers such that the sum of the numbers on her list is equal to the sum of the squares of the numbers on her list. Compute the largest possible value of the arithmetic mean of her numbers.

Proposed by Michael Ren

4 Compute the largest integer that can be expressed in the form $3^{x(3-x)}$ for some real number x.

[i]Proposed by James Lin

5 In triangle ABC, $AB = 8, AC = 9$, and $BC = 10$. Let M be the midpoint of BC. Circle ω_1 with area A_1 passes through A, B, and C. Circle ω_2 with area A_2 passes through A, B, and M. Then $\frac{A_1}{A_2} = \frac{m}{n}$ for relatively prime positive integers m and n. Compute $100m + n$.

Proposed by Luke Robitaille

6 Patchouli is taking an exam with $k > 1$ parts, numbered Part $1, 2, \ldots, k$. It is known that for $i = 1, 2, \ldots, k$, Part i contains i multiple choice questions, each of which has $(i + 1)$ answer choices. It is known that if she guesses randomly on every single question, the probability that she gets exactly one question correct is equal to 2018 times the probability that she gets no questions correct. Compute the number of questions that are on the exam.

Proposed by Yannick Yao

7 Compute the number of ways to erase 24 letters from the string "OMOMO$\cdots$OMO" (with length 27), such that the three remaining letters are O, M and O in that order. Note that the order in which they are erased does not matter.

[i]Proposed by Yannick Yao

8 Let ABC be the triangle with vertices located at the center of masses of Vincent Huang's house, Tristan Shin's house, and Edward Wan's house; here, assume the three are not collinear. Let $N = 2017$, and define the A-*ntipodes* to be the points $A_1, \ldots, A_N$ to be the points on segment BC such that $BA_1 = A_1A_2 = \cdots = A_{N-1}A_N = A_NC$, and similarly define the B, C-ntipodes. A line ℓ_A through A is called a *qevian* if it passes through an A-ntipode, and similarly we define qevians through B and C. Compute the number of ordered triples (ℓ_A, ℓ_B, ℓ_C) of concurrent qevians through A, B, C, respectively.

Proposed by Brandon Wang

9 Ann and Drew have purchased a mysterious slot machine; each time it is spun, it chooses a random positive integer such that k is chosen with probability 2^{-k} for every positive integer k, and then it outputs k tokens. Let N be a fixed integer. Ann and Drew alternate turns spinning the machine, with Ann going first. Ann wins if she receives at least N total tokens from the slot machine before Drew receives at least $M = 2^{2018}$ total tokens, and Drew wins if he receives M tokens before Ann receives N tokens. If each person has the same probability of winning, compute the remainder when N is divided by 2018.

Proposed by Brandon Wang

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10 Compute the largest prime factor of $357! + 358! + 359! + 360!$.

[i]Proposed by Luke Robitaille

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11 Let an ordered pair of positive integers (m, n) be called *regimented* if for all nonnegative integers k, the numbers m^k and n^k have the same number of positive integer divisors. Let N be the smallest positive integer such that $\left(2016^{2016}, N\right)$ is regimented. Compute the largest positive integer v such that 2^v divides the difference $2016^{2016} - N$.

Proposed by Ashwin Sah

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12 Three non-collinear lattice points A, B, C lie on the plane $1 + 3x + 5y + 7z = 0$. The minimal possible area of triangle ABC can be expressed as $\frac{\sqrt{m}}{n}$ where m, n are positive integers such that there does not exists a prime p dividing n with p^2 dividing m. Compute $100m + n$.

Proposed by Yannick Yao

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13 Compute the largest possible number of distinct real solutions for x to the equation

$$x^6 + ax^5 + 60x^4 - 159x^3 + 240x^2 + bx + c = 0,$$

where a, b, and c are real numbers.

[i]Proposed by Tristan Shin

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14 In triangle ABC, $AB = 13, BC = 14, CA = 15$. Let Ω and ω be the circumcircle and incircle of ABC respectively. Among all circles that are tangent to both Ω and ω, call those that contain ω *inclusive* and those that do not contain ω *exclusive*. Let $\mathcal{I}$ and $\mathcal{E}$ denote the set of centers of inclusive circles and exclusive circles respectively, and let I and E be the area of the regions enclosed by $\mathcal{I}$ and $\mathcal{E}$ respectively. The ratio $\frac{I}{E}$ can be expressed as $\sqrt{\frac{m}{n}}$, where m and n are relatively prime positive integers. Compute $100m + n$.

Proposed by Yannick Yao

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15 Iris does not know what to do with her 1-kilogram pie, so she decides to share it with her friend Rosabel. Starting with Iris, they take turns to give exactly half of total amount of pie (by mass) they possess to the other person. Since both of them prefer to have as few number of pieces of pie as possible, they use the following strategy: During each person's turn, she orders the pieces of pie that she has in a line from left to right in increasing order by mass, and starts giving the pieces of pie to the other person beginning from the left. If she encounters a piece that exceeds the remaining mass to give, she cuts it up into two pieces with her sword and gives the appropriately sized piece to the other person.

When the pie has been cut into a total of 2017 pieces, the largest piece that Iris has is $\frac{m}{n}$ kilograms, and the largest piece that Rosabel has is $\frac{p}{q}$ kilograms, where m, n, p, q are positive integers satisfying $\gcd(m, n) = \gcd(p, q) = 1$. Compute the remainder when $m + n + p + q$ is divided by 2017.

Proposed by Yannick Yao

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16 Jay has a 24×24 grid of lights, all of which are initially off. Each of the 48 rows and columns has a switch that toggles all the lights in that row and column, respectively, i.e. it switches lights that are on to off and lights that are off to on. Jay toggles each of the 48 rows and columns exactly once, such that after each toggle he waits for one minute before the next toggle. Each light uses no energy while off and 1 kiloJoule of energy per minute while on. To express his creativity, Jay chooses to toggle the rows and columns in a random order. Compute the expected value of the total amount of energy in kiloJoules which has been expended by all the lights after all 48 toggles.

[i]Proposed by James Lin

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17 A hyperbola in the coordinate plane passing through the points $(2, 5)$, $(7, 3)$, $(1, 1)$, and $(10, 10)$ has an asymptote of slope $\frac{20}{17}$. The slope of its other asymptote can be expressed in the form $-\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $100m + n$.

Proposed by Michael Ren

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18 On Lineland there are 2018 bus stations numbered 1 through 2018 from left to right. A self-driving bus that can carry at most N passengers starts from station 1 and drives all the way to station 2018, while making a stop at each bus station. Each passenger that gets on the bus at station i will get off at station j for some $j > i$ (the value of j may vary over different passengers). Call any group of four distinct stations i_1, i_2, j_1, j_2 with $i_u < j_v$ for all $u, v \in \{1, 2\}$ a *good* group. Suppose that in any good group i_1, i_2, j_1, j_2, there is a passenger who boards at station i_1 and de-boards at station j_1, or there is a passenger who boards at station i_2 and de-boards at station j_2, or both scenarios occur. Compute the minimum possible value of N.

Proposed by Yannick Yao

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19 Players $1, 2, \ldots, 10$ are playing a game on Christmas. Santa visits each player's house according to a set of rules:

-Santa first visits player 1. After visiting player i, Santa visits player $i+1$, where player 11 is the same as player 1.
-Every time Santa visits someone, he gives them either a present or a piece of coal (but not both).

-The absolute difference between the number of presents and pieces of coal that Santa has given out is at most 3 at every point in time.
-If Santa has a choice between giving out a present and a piece of coal, he chooses with equal probability.

Let p be the probability that player 1 gets a present before player 2 does. If $p = \frac{m}{n}$ for relatively prime positive integers m and n, then compute $100m + n$.

[i]Proposed by Tristan Shin

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20 For positive integers k, n with $k \le n$, we say that a k-tuple $(a_1, a_2, \ldots, a_k)$ of positive integers is *tasty* if

- there exists a k-element subset S of $[n]$ and a bijection $f : [k] \to S$ with $a_x \le f(x)$ for each $x \in [k]$,
- $a_x = a_y$ for some distinct $x, y \in [k]$, and
- $a_i \le a_j$ for any $i < j$.

For some positive integer n, there are more than 2018 tasty tuples as k ranges through $2, 3, \ldots, n$. Compute the least possible number of tasty tuples there can be.

Note: For a positive integer m, $[m]$ is taken to denote the set $\{1, 2, \ldots, m\}$.

Proposed by Vincent Huang and Tristan Shin

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21 Suppose that a sequence $a_0, a_1, \ldots$ of real numbers is defined by $a_0 = 1$ and

$$a_n = \begin{cases} a_{n-1}a_0 + a_{n-3}a_2 + \cdots + a_0a_{n-1} & \text{if } n \text{ odd} \\ a_{n-1}a_1 + a_{n-3}a_3 + \cdots + a_1a_{n-1} & \text{if } n \text{ even} \end{cases}$$

for $n \ge 1$. There is a positive real number r such that

$$a_0 + a_1r + a_2r^2 + a_3r^3 + \cdots = \frac{5}{4}.$$

If r can be written in the form $\frac{a\sqrt{b}-c}{d}$ for positive integers a, b, c, d such that b is not divisible by the square of any prime and $\gcd(a, c, d) = 1$, then compute $a + b + c + d$.

Proposed by Tristan Shin

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22 Let ABC be a triangle with $AB = 2$ and $AC = 3$. Let H be the orthocenter, and let M be the midpoint of BC. Let the line through H perpendicular to line AM intersect line AB at X and line AC at Y. Suppose that lines BY and CX are parallel. Then $[ABC]^2 = \frac{a+b\sqrt{c}}{d}$ for positive integers a, b, c and d, where $\gcd(a, b, d) = 1$ and c is not divisible by the square of any prime. Compute $1000a + 100b + 10c + d$.

[i]Proposed by Luke Robitaille

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23 Consider all ordered pairs (a, b) of positive integers such that $\frac{a^2+b^2+2}{ab}$ is an integer and $a \le b$. We label all such pairs in increasing order by their distance from the origin. (It is guaranteed that no ties exist.) Thus $P_1 = (1,1), P_2 = (1,3)$, and so on. If $P_{2020} = (x, y)$, then compute the remainder when $x + y$ is divided by 2017.

Proposed by Ashwin Sah

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24 Let $p = 101$ and let S be the set of p-tuples $(a_1, a_2, \dots, a_p) \in \mathbb{Z}^p$ of integers. Let N denote the number of functions $f : S \to \{0, 1, \dots, p-1\}$ such that

- $f(a+b) + f(a-b) \equiv 2\big(f(a) + f(b)\big) \pmod{p}$ for all $a, b \in S$, and
- $f(a) = f(b)$ whenever all components of $a - b$ are divisible by p.

Compute the number of positive integer divisors of N. (Here addition and subtraction in $\mathbb{Z}^p$ are done component-wise.)

Proposed by Ankan Bhattacharya

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25 Given two positive integers x, y, we define $z = x \oplus y$ to be the bitwise XOR sum of x and y; that is, z has a 1 in its binary representation at exactly the place values where x, y have differing binary representations. It is known that $\oplus$ is both associative and commutative. For example, $20 \oplus 18 = 10100_2 \oplus 10010_2 = 110_2 = 6$. Given a set $S = \{a_1, a_2, \dots, a_n\}$ of positive integers, we let $f(S) = a_1 \oplus a_2 \oplus a_3 \oplus \cdots \oplus a_n$. We also let $g(S)$ be the number of divisors of $f(S)$ which are at most 2018 but greater than or equal to the largest element in S (if S is empty then let $g(S) = 2018$). Compute the number of 1s in the binary representation of $\displaystyle\sum_{S \subseteq \{1,2,\dots,2018\}} g(S)$.

[i]Proposed by Brandon Wang and Vincent Huang

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26 Let $p = 2027$ be the smallest prime greater than 2018, and let $P(X) = X^{2031} + X^{2030} + X^{2029} - X^5 - 10X^4 - 10X^3 + 2018X^2$. Let $\mathrm{GF}(p)$ be the integers modulo p, and let $\mathrm{GF}(p)(X)$ be the set of rational functions with coefficients in $\mathrm{GF}(p)$ (so that all coefficients are taken modulo p). That is, $\mathrm{GF}(p)(X)$ is the set of fractions $\frac{P(X)}{Q(X)}$ of polynomials with coefficients in $\mathrm{GF}(p)$, where $Q(X)$ is not the zero polynomial. Let $D\colon \mathrm{GF}(p)(X) \to \mathrm{GF}(p)(X)$ be a function satisfying

$$D\left(\frac{f}{g}\right) = \frac{D(f)\cdot g - f \cdot D(g)}{g^2}$$

for any $f, g \in \mathrm{GF}(p)(X)$ with $g \neq 0$, and such that for any nonconstant polynomial f, $D(f)$ is a polynomial with degree less than that of f. If the number of possible values of $D(P(X))$ can be written as a^b, where a, b are positive integers with a minimized, compute ab.

Proposed by Brandon Wang

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27 Let $p = 2^{16} + 1$ be a prime. Let N be the number of ordered tuples (A, B, C, D, E, F) of integers between 0 and $p-1$, inclusive, such that there exist integers x, y, z not all divisible by p with

p dividing all three of $Ax + Ez + Fy, By + Dz + Fx, Cz + Dy + Ex$. Compute the remainder when N is divided by 10^6.

Proposed by Vincent Huang

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28 Let ω be a circle centered at O with radius $R = 2018$. For any $0 < r < 1009$, let γ be a circle of radius r centered at a point I satisfying $OI = \sqrt{R(R-2r)}$. Choose any $A, B, C \in \omega$ with AC, AB tangent to γ at E, F, respectively. Suppose a circle of radius r_A is tangent to AB, AC, and internally tangent to ω at a point D with $r_A = 5r$. Let line EF meet ω at P_1, Q_1. Suppose P_2, P_3, Q_2, Q_3 lie on ω such that $P_1P_2, P_1P_3, Q_1Q_2, Q_1Q_3$ are tangent to γ. Let P_2P_3, Q_2Q_3 meet at K, and suppose KI meets AD at a point X. Then as r varies from 0 to 1009, the maximum possible value of OX can be expressed in the form $\frac{a\sqrt{b}}{c}$, where a, b, c are positive integers such that b is not divisible by the square of any prime and $\gcd(a, c) = 1$. Compute $10a + b + c$.

[i]Proposed by Vincent Huang

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29 For integers $0 \le m, n \le 2^{2017} - 1$, let $\alpha(m, n)$ be the number of nonnegative integers k for which $\lfloor m/2^k \rfloor$ and $\lfloor n/2^k \rfloor$ are both odd integers. Consider a $2^{2017} \times 2^{2017}$ matrix M whose (i, j)th entry (for $1 \le i, j \le 2^{2017}$) is

$$(-1)^{\alpha(i-1, j-1)}.$$

For $1 \le i, j \le 2^{2017}$, let $M_{i,j}$ be the matrix with the same entries as M except for the (i, j)th entry, denoted by $a_{i,j}$, and such that $\det M_{i,j} = 0$. Suppose that A is the $2^{2017} \times 2^{2017}$ matrix whose (i, j)th entry is $a_{i,j}$ for all $1 \le i, j \le 2^{2017}$. Compute the remainder when $\det A$ is divided by 2017.

Proposed by Michael Ren and Ashwin Sah

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30 Let ABC be an acute triangle with $\cos B = \frac{1}{3}, \cos C = \frac{1}{4}$, and circumradius 72. Let ABC have circumcenter O, symmedian point K, and nine-point center N. Consider all non-degenerate hyperbolas $\mathcal{H}$ with perpendicular asymptotes passing through A, B, C. Of these $\mathcal{H}$, exactly one has the property that there exists a point $P \in \mathcal{H}$ such that NP is tangent to $\mathcal{H}$ and $P \in OK$. Let N' be the reflection of N over BC. If AK meets PN' at Q, then the length of PQ can be expressed in the form $a + b\sqrt{c}$, where a, b, c are positive integers such that c is not divisible by the square of any prime. Compute $100a + b + c$.

Proposed by Vincent Huang
