Math Prize for Girls Problems 2018

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by Ravi B

1 If x is a real number such that $(x-3)(x-1)(x+1)(x+3)+16 = 116^2$, what is the largest possible value of x?

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2 How many ordered pairs of integers (x, y) satisfy $2|y| \le x \le 40$?

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3 Let S be the set of all positive integers from 1 through 1000 that are not perfect squares. What is the length of the longest, non-constant, arithmetic sequence that consists of elements of S?

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4 Let $ABCDEF$ be a regular hexagon. Let P be the intersection point of $\overline{AC}$ and $\overline{BD}$. Suppose that the area of triangle EFP is 25. What is the area of the hexagon?

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5 Consider the following system of 7 linear equations with 7 unknowns:

$$\begin{aligned} a+b+c+d+e &= 1 \\ b+c+d+e+f &= 2 \\ c+d+e+f+g &= 3 \\ d+e+f+g+a &= 4 \\ e+f+g+a+b &= 5 \\ f+g+a+b+c &= 6 \\ g+a+b+c+d &= 7. \end{aligned}$$

What is g?

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6 Martha writes down a random mathematical expression consisting of 3 single-digit positive integers with an addition sign "$+$" or a multiplication sign "$\times$" between each pair of adjacent digits. (For example, her expression could be $4+3\times 3$, with value 13.) Each positive digit is equally likely, each arithmetic sign ("$+$" or "$\times$") is equally likely, and all choices are independent. What is the expected value (average value) of her expression?

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7 For every positive integer n, let $T_n = \frac{n(n+1)}{2}$ be the n^{th} triangular number. What is the 2018^{th} smallest positive integer n such that T_n is a multiple of 1000?

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8 A mustache is created by taking the set of points (x, y) in the xy-coordinate plane that satisfy $4 + 4\cos(\pi x/24) \le y \le 6 + 6\cos(\pi x/24)$ and $-24 \le x \le 24$. What is the area of the mustache?

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9 How many 3-term geometric sequences a, b, c are there where a, b, and c are positive integers with $a < b < c$ and $c = 8000$?

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10 Let T_1 be an isosceles triangle with sides of length 8, 11, and 11. Let T_2 be an isosceles triangle with sides of length b, 1, and 1. Suppose that the radius of the incircle of T_1 divided by the radius of the circumcircle of T_1 is equal to the radius of the incircle of T_2 divided by the radius of the circumcircle of T_2. Determine the largest possible value of b.

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11 Maryam has a fair tetrahedral die, with the four faces of the die labeled 1 through 4. At each step, she rolls the die and records which number is on the bottom face. She stops when the current number is greater than or equal to the previous number. (In particular, she takes at least two steps.) What is the expected number (average number) of steps that she takes?

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12 You own a calculator that computes exactly. It has all the standard buttons, including a button that replaces the number currently displayed with its arctangent, and a button that replaces whatever is currently displayed with its cosine. You turn on the calculator and it reads 0. You create a sequence by alternately clicking on the arctangent button and the cosine button. (The calculator is in radian mode.) Let a_n be the value displayed after you've pressed the cosine button for the nth time. What is $\prod_{k=1}^{11} a_k$?

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13 A circle overlaps an equilateral triangle of side length $100\sqrt{3}$. The three chords in the circle formed by the three sides of the triangle have lengths 6, 36, and 60, respectively. What is the area of the circle?

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14 Let $f(x)$ be the polynomial $\prod_{k=1}^{50}\big(x - (2k-1)\big)$. Let c be the coefficient of x^{48} in $f(x)$. When c is divided by 101, what is the remainder? (The remainder is an integer between 0 and 100.)

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15 In the xy-coordinate plane, the x-axis and the line $y = x$ are mirrors. If you shoot a laser beam from the point $(126, 21)$ toward a point on the positive x-axis, there are 3 places you can aim at where the beam will bounce off the mirrors and eventually return to $(126, 21)$. They are $(126, 0)$, $(105, 0)$, and a third point $(d, 0)$. What is d? (Recall that when light bounces off a mirror, the angle of incidence has the same measure as the angle of reflection.)

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16 Define a function f on the unit interval $0 \le x \le 1$ by the rule

$$f(x) = \begin{cases} 1 - 3x & \text{if } 0 \le x < 1/3\,; \\ 3x - 1 & \text{if } 1/3 \le x < 2/3\,; \\ 3 - 3x & \text{if } 2/3 \le x \le 1\,. \end{cases}$$

Determine $f^{(2018)}(1/730)$. Recall that $f^{(n)}$ denotes the nth iterate of f; for example, $f^{(3)}(1/730) = f(f(f(1/730)))$.

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17 Let ABC be a triangle with $AB = 5$, $BC = 4$, and $CA = 3$. On each side of ABC, externally erect a semicircle whose diameter is the corresponding side. Let X be on the semicircular arc erected on side $\overline{BC}$ such that $\angle CBX$ has measure 15°. Let Y be on the semicircular arc erected on side $\overline{CA}$ such that $\angle ACY$ has measure 15°. Similarly, let Z be on the semicircular arc erected on side $\overline{AB}$ such that $\angle BAZ$ has measure 15°. What is the area of triangle XYZ?

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18 Evaluate the expression

$$\left| \prod_{k=0}^{15} \left(1 + e^{2\pi i k^2/31}\right) \right| .$$

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19 Consider the sum

$$S_n = \sum_{k=1}^{n} \frac{1}{\sqrt{2k-1}} .$$

Determine $\lfloor S_{4901} \rfloor$. Recall that if x is a real number, then $\lfloor x \rfloor$ (the *floor* of x) is the greatest integer that is less than or equal to x.

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20 A *smooth number* is a positive integer of the form $2^m 3^n$, where m and n are nonnegative integers. Let S be the set of all triples (a, b, c) where a, b, and c are smooth numbers such that $\gcd(a, b)$, $\gcd(b, c)$, and $\gcd(c, a)$ are all distinct. Evaluate the infinite sum $\sum_{(a,b,c)\in S} \frac{1}{abc}$. Recall that $\gcd(x, y)$ is the greatest common divisor of x and y.
