Math Prize for Girls Olympiad 2014

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by Ravi B

1 Say that a convex quadrilateral is *tasty* if its two diagonals divide the quadrilateral into four nonoverlapping similar triangles. Find all tasty convex quadrilaterals. Justify your answer.

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2 Let f be the function defined by $f(x) = 4x(1-x)$. Let n be a positive integer. Prove that there exist distinct real numbers $x_1, x_2, \ldots, x_n$ such that $x_{i+1} = f(x_i)$ for each integer i with $1 \le i \le n-1$, and such that $x_1 = f(x_n)$.

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3 Say that a positive integer is *sweet* if it uses only the digits 0, 1, 2, 4, and 8. For instance, 2014 is sweet. There are sweet integers whose squares are sweet: some examples (not necessarily the smallest) are 1, 2, 11, 12, 20, 100, 202, and 210. There are sweet integers whose cubes are sweet: some examples (not necessarily the smallest) are 1, 2, 10, 20, 200, 202, 281, and 2424. Prove that there exists a sweet positive integer n whose square and cube are both sweet, such that the sum of all the digits of n is 2014.

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4 Let n be a positive integer. A 4-by-n rectangle is divided into $4n$ unit squares in the usual way. Each unit square is colored black or white. Suppose that every white unit square shares an edge with at least one black unit square. Prove that there are at least n black unit squares.
