HMMT Invitational Competition 2015

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by va2010, math154

1

Let S be the set of positive integers n such that the inequality

$$\phi(n) \cdot \tau(n) \geq \sqrt{\frac{n^3}{3}}$$

holds, where $\phi(n)$ is the number of positive integers $k \leq n$ that are relatively prime to n, and $\tau(n)$ is the number of positive divisors of n. Prove that S is finite.

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2 Let m, n be positive integers with $m \geq n$. Let S be the set of pairs (a, b) of relatively prime positive integers such that $a, b \leq m$ and $a + b > m$.

For each pair $(a, b) \in S$, consider the nonnegative integer solution (u, v) to the equation $au - bv = n$ chosen with $v \geq 0$ minimal, and let $I(a, b)$ denote the (open) interval $(v/a, u/b)$.

Prove that $I(a, b) \subseteq (0, 1)$ for every $(a, b) \in S$, and that any fixed irrational number $\alpha \in (0, 1)$ lies in $I(a, b)$ for exactly n distinct pairs $(a, b) \in S$.

Victor Wang, inspired by 2013 ISL N7

3

Let M be a 2014×2014 invertible matrix, and let $\mathcal{F}(M)$ denote the set of matrices whose rows are a permutation of the rows of M. Find the number of matrices $F \in \mathcal{F}(M)$ such that $\det(M + F) \neq 0$.

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4 Prove that there exists a positive integer N such that for any positive integer $n \geq N$, there are at least 2015 non-empty subsets S of $\{n^2 + 1, n^2 + 2, \ldots, n^2 + 3n\}$ with the property that the product of the elements of S is a perfect square.

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5 Let $\omega = e^{2\pi i/5}$ be a primitive fifth root of unity. Prove that there do not exist integers a, b, c, d, k with $k > 1$ such that

$$(a + b\omega + c\omega^2 + d\omega^3)^k = 1 + \omega.$$

Carl Lian
