HMMT Invitational Competition 2014

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by Wave-Particle

1 Consider a regular n-gon with $n > 3$, call a line *acceptable* if it passes through the interior of this n-gon. Draw m different acceptable lines, so that the n-gon is divided into several smaller polygons.

(a) Prove that there exists an m, depending only on n, such that any collection of m acceptable lines results in one of the smaller polygons having 3 or 4 sides.

(b) Find the smallest possible m which guarantees that at least one of the smaller polygons will have 3 or 4 sides.

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2 2014 triangles have non-overlapping interiors contained in a circle of radius 1. What is the largest possible value of the sum of their areas?

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3 Fix positive integers m and n. Suppose that $a_1, a_2, \dots, a_m$ are reals, and that pairwise distinct vectors $v_1, \dots, v_m \in \mathbb{R}^n$ satisfy

$$\sum_{j \neq i} a_j \frac{v_j - v_i}{||v_j - v_i||^3} = 0$$

for $i = 1, 2, \dots, m$.

Prove that

$$\sum_{1 \leq i < j \leq m} \frac{a_i a_j}{||v_j - v_i||} = 0.$$

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4 Let ω be a root of unity and f be a polynomial with integer coefficients. Show that if $|f(\omega)| = 1$, then $f(\omega)$ is also a root of unity.

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5 Let n be a positive integer, and let A and B be $n \times n$ matrices with complex entries such that $A^2 = B^2$. Show that there exists an $n \times n$ invertible matrix S with complex entries that satisfies $S(AB - BA) = (BA - AB)S$.
