IberoAmerican 2019

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– Day 1

1 For each positive integer n, let $s(n)$ be the sum of the squares of the digits of n. For example, $s(15) = 1^2 + 5^2 = 26$. Determine all integers $n \geq 1$ such that $s(n) = n$.

2 Determine all polynomials $P(x)$ with degree $n \geq 1$ and integer coefficients so that for every real number x the following condition is satisfied

$$P(x) = (x - P(0))(x - P(1))(x - P(2)) \cdots (x - P(n-1))$$

3 Let Γ be the circumcircle of triangle ABC. The line parallel to AC passing through B meets Γ at D ($D \neq B$), and the line parallel to AB passing through C intersects Γ to E ($E \neq C$). Lines AB and CD meet at P, and lines AC and BE meet at Q. Let M be the midpoint of DE. Line AM meets Γ at Y ($Y \neq A$) and line PQ at J. Line PQ intersects the circumcircle of triangle BCJ at Z ($Z \neq J$). If lines BQ and CP meet each other at X, show that X lies on the line YZ.

– Day 2

4 Let $ABCD$ be a trapezoid with $AB \parallel CD$ and inscribed in a circumference Γ. Let P and Q be two points on segment AB (A, P, Q, B appear in that order and are distinct) such that $AP = QB$. Let E and F be the second intersection points of lines CP and CQ with Γ, respectively. Lines AB and EF intersect at G. Prove that line DG is tangent to Γ.

5 Don Miguel places a token in one of the $(n+1)^2$ vertices determined by an $n \times n$ board. A *move* consists of moving the token from the vertex on which it is placed to an adjacent vertex which is at most $\sqrt{2}$ away, as long as it stays on the board. A *path* is a sequence of moves such that the token was in each one of the $(n+1)^2$ vertices exactly once. What is the maximum number of diagonal moves (those of length $\sqrt{2}$) that a path can have in total?

6 Let $a_1, a_2, \dots, a_{2019}$ be positive integers and P a polynomial with integer coefficients such that, for every positive integer n,

$$P(n) \text{ divides } a_1^n + a_2^n + \cdots + a_{2019}^n.$$

Prove that P is a constant polynomial.

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