Princeton University Math Competition 2015

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by parmenides51, Vfire, AlcumusGuy, djmathman

– Geometry

A1/B3 For her daughters 12th birthday, Ingrid decides to bake a dodecagon pie in celebration. Unfortunately, the store does not sell dodecagon shaped pie pans, so Ingrid bakes a circular pie first and then trims off the sides in a way such that she gets the largest regular dodecagon possible. If the original pie was 8 inches in diameter, the area of pie that she has to trim off can be represented in square inches as $a\pi - b$ where a, b are integers. What is $a + b$?

A2/B4 Terry the Tiger lives on a cube-shaped world with edge length 2. Thus he walks on the outer surface. He is tied, with a leash of length 2, to a post located at the center of one of the faces of the cube. The surface area of the region that Terry can roam on the cube can be represented as $\frac{p\pi}{q} + a\sqrt{b} + c$ for integers a, b, c, p, q where no integer square greater than 1 divides b, p and q are coprime, and $q > 0$. What is $p + q + a + b + c$? (Terry can be at a location if the shortest distance along the surface of the cube between that point and the post is less than or equal to 2.)

A3/B5 Cyclic quadrilateral $ABCD$ satisfies $\angle ADC = 2 \cdot \angle BAD = 80^\circ$ and $\overline{BC} = \overline{CD}$. Let the angle bisector of $\angle BCD$ meet AD at P. What is the measure, in degrees, of $\angle BPD$?

A4/B6 Find the largest r such that 4 balls each of radius r can be packed into a regular tetrahedron with side length 1. In a packing, each ball lies outside every other ball, and every ball lies inside the boundaries of the tetrahedron. If r can be expressed in the form $\frac{\sqrt{a}+b}{c}$ where a, b, c are integers such that $\gcd(b, c) = 1$, what is $a + b + c$?

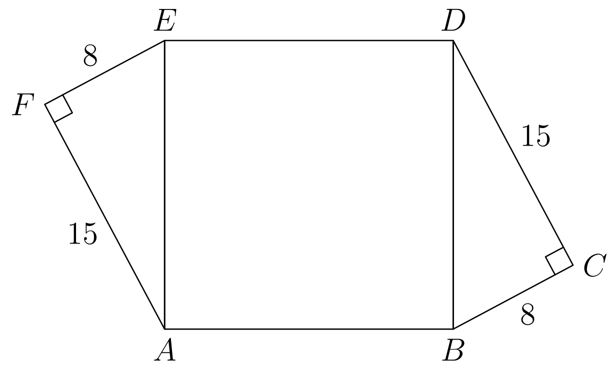
A5/B7 Let P, A, B, C be points on circle O such that C does not lie on arc $\widehat{BAP}$, $\overline{PA} = 21$, $\overline{PB} = 56$, $\overline{PC} = 35$ and $m\angle BPC = 60^\circ$. Now choose point D on the circle such that C does not lie on arc $\widehat{BDP}$ and $\overline{BD} = 39$. What is AD?

A6/B8 Triangle ABC is inscribed in a unit circle ω. Let H be its orthocenter and D be the foot of the perpendicular from A to BC. Let $\triangle XYZ$ be the triangle formed by drawing the tangents to ω at A, B, C. If $\overline{AH} = \overline{HD}$ and the side lengths of $\triangle XYZ$ form an arithmetic sequence, the area of $\triangle ABC$ can be expressed in the form $\frac{p}{q}$ for relatively prime positive integers p, q. What is $p + q$?

A7 Triangle ABC has $\overline{AB} = \overline{AC} = 20$ and $\overline{BC} = 15$. Let D be the point in $\triangle ABC$ such that $\triangle ADB \sim \triangle BDC$. Let l be a line through A and let BD and CD intersect l at P and Q, respectively. Let the circumcircles of $\triangle BDQ$ and $\triangle CDP$ intersect at X. The area of the locus of X as l varies can be expressed in the form $\frac{p}{q}\pi$ for positive coprime integers p and q. What is $p+q$?

A8 The incircle of acute triangle ABC touches BC, AC, and AB at points D, E, and F, respectively. Let P be the second intersection of line AD and the incircle. The line through P tangent to the incircle intersects AB and AC at points M and N, respectively. Given that $\overline{AB} = 8, \overline{AC} = 10$, and $\overline{AN} = 4$, let $\overline{AM} = \frac{a}{b}$ where a and b are positive coprime integers. What is $a + b$?

B1 Find the distance $\overline{CF}$ in the diagram below where $ABDE$ is a square and angles and lengths are as given:



The length $\overline{CF}$ is of the form $a\sqrt{b}$ for integers a, b such that no integer square greater than 1 divides b. What is $a + b$?

B2 Let $ABCD$ be a regular tetrahedron with side length 1. Let $EFGH$ be another regular tetrahedron such that the volume of $EFGH$ is $\frac{1}{8}$-th the volume of $ABCD$. The height of $EFGH$ (the minimum distance from any of the vertices to its opposing face) can be written as $\sqrt{\frac{a}{b}}$, where a and b are positive coprime integers. What is $a + b$?

– Number Theory

A1/B2 What is the 22nd positive integer n such that 22^n ends in a 2? (when written in base 10).

A2/B3 What is the sum of all positive integers n such that $\text{lcm}(2n, n^2) = 14n - 24$?

A3/B6 What is the largest positive integer n less than $10,000$ such that in base 4, n and $3n$ have the same number of digits; in base 8, n and $7n$ have the same number of digits; and in base 16, n and $15n$ have the same number of digits? Express your answer in base 10.

A4/B7 What is the smallest positive integer n such that $20 \equiv n^{15} \pmod{29}$?

A5 Given that there are 24 primes between 3 and 100, inclusive, what is the number of ordered pairs (p, a) with p prime, $3 \le p < 100$, and $1 \le a < p$ such that the sum

$$a + a^2 + a^3 + \cdots + a^{(p-2)!}$$

is not divisible by p?

A6 For a positive integer n, let $d(n)$ be the number of positive divisors of n. What is the smallest positive integer n such that

$$\sum_{t|n} d(t)^3$$

is divisible by 35?

A7/B8 Given a positive integer k, let $f(k)$ be the sum of the k-th powers of the primitive roots of 73. For how many positive integers $k < 2015$ is $f(k)$ divisible by $73?$

[i]Note: A primitive root of r of a prime p is an integer $1 \le r < p$ such that the smallest positive integer k such that $r^k \equiv 1 \pmod{p}$ is $k = p - 1$.[/i]

A8 Let $n = 2^{2015} - 1$. For any integer $1 \le x < n$, let

$$f_n(x) = \sum_p s_p(n-x) + s_p(x) - s_p(n),$$

where $s_q(k)$ denotes the sum of the digits of k when written in base q and the summation is over all primes p. Let N be the number of values of x such that $4|f_n(x)$. What is the remainder when N is divided by $1000?$

B1 What is the remainder when

$$\sum_{k=0}^{100} 10^k$$

is divided by 9?

B4 A circle with radius 1 and center $(0, 1)$ lies on the coordinate plane. Ariel stands at the origin and rolls a ball of paint at an angle of 35 degrees relative to the positive x-axis (counting degrees counterclockwise). The ball repeatedly bounces off the circle and leaves behind a trail of paint where it rolled. After the ball of paint returns to the origin, the paint has traced out a star with n points on the circle. What is n?

B5 Given that there are 24 primes between 3 and 100, inclusive, what is the number of ordered pairs (p, a) with p prime, $3 \le p < 100$, and $1 \le a < p$ such that $p \mid (a^{p-2} - a)$?

– Combinatorics

A1/B1 A word is an ordered, non-empty sequence of letters, such as $word$ or $wrod$. How many distinct 3-letter words can be made from a subset of the letters c, o, m, b, o, where each letter in the list is used no more than the number of times it appears?

A2/B4 Andrew has 10 balls in a bag, each a different color. He randomly picks a ball from the bag 4 times, with replacement. The expected number of distinct colors among the balls he picks is $\frac{p}{q}$, where $\gcd(p, q) = 1$ and $p, q > 0$. What is $p + q$?

A3/B5 Consider a random permutation of the set $\{1, 2, ..., 2015\}$. In other words, for each $1 \leq i \leq 2015$, i is sent to the element a_i where $a_i \in \{1, 2, ..., 2015\}$ and if $i \neq j$, then $a_i \neq a_j$. What is the expected number of ordered pairs (a_i, a_j) with $i - j > 155$ and $a_i - a_j > 266$?

A4/B6 A number is *interesting* if it is a 6-digit integer that contains no zeros, its first 3 digits are strictly increasing, and its last 3 digits are non-increasing. What is the average of all interesting numbers?

A5/B7 Alice has an orange 3-by-3-by-3 cube, which is comprised of 27 distinguishable, 1-by-1-by-1 cubes. Each small cube was initially orange, but Alice painted 10 of the small cubes completely black. In how many ways could she have chosen 10 of these smaller cubes to paint black such that every one of the 27 3-by-1-by-1 sub-blocks of the 3-by-3-by-3 cube contains at least one small black cube?

A6 Every day, Heesu talks to Sally with some probability p. One day, after not talking to Sally the previous day, Heesu resolves to ask Sally out on a date. From now on, each day, if Heesu has talked to Sally each of the past four days, then Heesu will ask Sally out on a date. Heesus friend remarked that at this rate, it would take Heesu an expected 2800 days to finally ask Sally out. Suppose $p = \frac{m}{n}$, where $\gcd(m, n) = 1$ and $m, n > 0$. What is $m + n$?

A7 The lattice points (i, j) for integers $0 \leq i, j \leq 3$ are each being painted orange or black. Suppose a coloring is good if for every set of integers x_1, x_2, y_1, y_2 such that $0 \leq x_1 < x_2 \leq 3$ and $0 \leq y_1 < y_2 \leq 3$, the points $(x_1, y_1), (x_1, y_2), (x_2, y_1), (x_2, y_2)$ are not all the same color. How many good colorings are possible?

A8 In a tournament with 2015 teams, each team plays every other team exactly once and no ties occur. Such a tournament is *imbalanced* if for every group of 6 teams, there exists either a team that wins against the other 5 or a team that loses to the other 5. If the teams are indistinguishable, what is the number of distinct imbalanced tournaments that can occur?

B2 Jonathan has a magical coin machine which takes coins in amounts of $7, 8$, and 9. If he puts in 7 coins, he gets 3 coins back; if he puts in 8, he gets 11 back; and if he puts in 9, he gets 4 back. The coin machine does not allow two entries of the same amount to happen consecutively.

Starting with 15 coins, what is the minimum number of entries he can make to end up with 4 coins?

B3 Princetons Math Club recently bought a stock for $\$2$ and sold it for $\$9$ thirteen days later. Given that the stock either increases or decreases by $\$1$ every day and never reached $\$0$, in how many possible ways could the stock have changed during those thirteen days?

B8 In how many ways can 9 cells of a 6-by-6 grid be painted black such that no two black cells share a corner or an edge with each other?

– Algebra

A1 How many integer pairs (a, b) with $1 < a, b \leq 2015$ are there such that $\log_a b$ is an integer?

A2/B4 There are real numbers a, b, c, d such that for all (x, y) satisfying $6y^2 = 2x^3 + 3x^2 + x$, if $x_1 = ax + b$ and $y_1 = cy + d$, then $y_1^2 = x_1^3 - 36x_1$. What is $a + b + c + d$?

A3/B5 Find the sum of the non-repeated roots of the polynomial $P(x) = x^6 - 5x^5 - 4x^4 - 5x^3 + 8x^2 + 7x + 7$.

A4/B6 Define the sequence a_i as follows: $a_1 = 1, a_2 = 2015$, and $a_n = \frac{na_{n-1}^2}{a_{n-1}+na_{n-2}}$ for $n > 2$. What is the least k such that $a_k < a_{k-1}$?

A5 Since counting the numbers from 1 to 100 wasn't enough to stymie Gauss, his teacher devised another clever problem that he was sure would stump Gauss. Defining $\zeta_{15} = e^{2\pi i/15}$ where $i = \sqrt{-1}$, the teacher wrote the 15 complex numbers ζ_{15}^k for integer $0 \leq k < 15$ on the board. Then, he told Gauss:

On every turn, erase two random numbers a, b, chosen uniformly randomly, from the board and then write the term $2ab - a - b + 1$ on the board instead. Repeat this until you have one number left. What is the expected value of the last number remaining on the board?

A6/B7 We define the function $f(x, y) = x^3 + (y - 4)x^2 + (y^2 - 4y + 4)x + (y^3 - 4y^2 + 4y)$. Then choose any distinct $a, b, c \in \mathbb{R}$ such that the following holds: $f(a, b) = f(b, c) = f(c, a)$. Over all such choices of a, b, c, what is the maximum value achieved by

$$\min(a^4 - 4a^3 + 4a^2, b^4 - 4b^3 + 4b^2, c^4 - 4c^3 + 4c^2)?$$

A7/B8 We define the ridiculous numbers recursively as follows:

-1 is a ridiculous number.
-If a is a ridiculous number, then $\sqrt{a}$ and $1 + \sqrt{a}$ are also ridiculous numbers.

A closed interval I is "boring" if

-I contains no ridiculous numbers, and
-There exists an interval $[b, c]$ containing I for which b and c are both ridiculous numbers.

The smallest non-negative l such that there does not exist a boring interval with length l can be represented in the form $\frac{a+b\sqrt{c}}{d}$ where a, b, c, d are integers, $\gcd(a, b, d) = 1$, and no integer square greater than 1 divides c. What is $a + b + c + d$?

A8 Let $P(x)$ be a polynomial with positive integer coefficients and degree 2015. Given that there exists some $\omega \in \mathbb{C}$ satisfying

$$\omega^{73} = 1 \quad \text{and}$$

$$P(\omega^{2015}) + P(\omega^{2015^2}) + P(\omega^{2015^3}) + \ldots + P(\omega^{2015^{72}}) = 0,$$

what is the minimum possible value of $P(1)$?

B1 Roy is starting a baking company and decides that he will sell cupcakes. He sells n cupcakes for $(n + 20)(n + 15)$ cents. A man walks in and buys $\$10.50$ worth of cupcakes. Roy bakes cupcakes at a rate of 10 cupcakes an hour. How many minutes will it take Roy to complete the order?

B2 Let f be a function which takes in $0, 1, 2$ and returns $0, 1,$ or 2. The values need not be distinct: for instance we could have $f(0) = 1, f(1) = 1, f(2) = 2$. How many such functions are there which satisfy

$$f(2) + f(f(0)) + f(f(f(1))) = 5?$$

B3 Andrew and Blair are bored in class and decide to play a game. They pick a pair (a, b) with $1 \le a, b \le 100$. Andrew says the next number in the geometric series that begins with a, b and Blair says the next number in the arithmetic series that begins with a, b. For how many pairs (a, b) is Andrew's number minus Blair's number a positive perfect square?

– Individual Finals

A1/B1 Alice places down n bishops on a 2015×2015 chessboard such that no two bishops are attacking each other. (Bishops attack each other if they are on a diagonal.)

-Find, with proof, the maximum possible value of n.
-(A1 only) For this maximal n, find, with proof, the number of ways she could place her bishops on the chessboard.

A2/B3 For an odd prime number p, let S denote the following sum taken modulo p:

$$S \equiv \frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{(p-2) \cdot (p-1)} \equiv \sum_{i=1}^{\frac{p-1}{2}} \frac{1}{(2i-1) \cdot 2i} \pmod p$$

Prove that $p^2 | 2^p - 2$ if and only if $S \equiv 0 \pmod p$.

A3 Let I be the incenter of a triangle ABC with $AB = 20$, $BC = 15$, and $BI = 12$. Let CI intersect the circumcircle ω_1 of ABC at $D \neq C$. Alice draws a line l through D that intersects ω_1 on the minor arc AC at X and the circumcircle ω_2 of AIC at Y outside ω_1. She notices that she can construct a right triangle with side lengths ID, DX, and XY. Determine, with proof, the length of IY.

B2 On a circle ω_1, four points A, C, B, D lie in that order. Prove that $CD^2 = AC \cdot BC + AD \cdot BD$ if and only if at least one of C and D is the midpoint of arc AB.

– Team

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1 Let $f(n)$ denote the sum of the distinct positive integer divisors of n. Evaluate:

$$f(1) + f(2) + f(3) + f(4) + f(5) + f(6) + f(7) + f(8) + f(9).$$

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2 Sally is going shopping for stuffed tigers. She finds 5 orange, 10 white, and 2 cinnamon colored tigers. Sally decides to buy two tigers of different colors. Assuming all the tigers are distinct, in how many ways can she choose two tigers?

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3 How many ordered pairs (a, b) of positive integers with $1 \leq a, b \leq 10$ are there such that in the geometric sequence whose first term is a and whose second term is b, the third term is an integer?

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4 Ryan is messing with Brices coin. He weights the coin such that it comes up on one side twice as frequently as the other, and he chooses whether to weight heads or tails more with equal probability. Brice flips his modified coin twice and it lands up heads both times. The probability that the coin lands up heads on the next flip can be expressed in the form $\frac{p}{q}$ for positive integers p, q satisfying $\gcd(p, q) = 1$, what is $p + q$?

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5 Imagine a regular a 2015-gon with edge length 2. At each vertex, draw a unit circle centered at that vertex and color the circles circumference orange. Now, another unit circle S is placed inside the polygon such that it is externally tangent to two adjacent circles centered at the vertices. This circle S is allowed to roll freely in the interior of the polygon as long as it remains externally tangent to the vertex circles. As it rolls, S turns the color of any point it touches into black. After it rolls completely around the interior of the polygon, the total length of the black lengths can be expressed in the form $\frac{p\pi}{q}$ for positive integers p, q satisfying $\gcd(p, q) = 1$. What is $p + q$?

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6 What is the smallest positive integer n such that $2^n - 1$ is a multiple of 2015?

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7 Charlie noticed his golden ticket was golden in two ways! In addition to being gold, it was a rectangle whose side lengths had ratio the golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$. He then folds the ticket so that two opposite corners (vertices connected by a diagonal) coincide and makes a sharp crease (the ticket folds just as any regular piece of paper would). The area of the resulting shape can be expressed as $a + b\varphi$. What is $\frac{b}{a}$?

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8 Let $\sigma_1 : \mathbb{N} \to \mathbb{N}$ be a function that takes a natural number n, and returns the sum of the positive integer divisors of n. For example, $\sigma_1(6) = 1 + 2 + 3 + 6 = 12$. What is the largest number n such that $\sigma_1(n) = 1854$?

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9 Triangle ABC has $\overline{AB} = 5, \overline{BC} = 4, \overline{CA} = 6$. Points D and E are on sides AB and AC, respectively, such that $\overline{AD} = \overline{AE} = \overline{BC}$. Let CD and BE intersect at F and let AF and DE intersect at G. The length of FG can be expressed in the form $\frac{a\sqrt{b}}{c}$ in simplified form. What is $a + b + c$?

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10 Let S be the set of integer triplets (a, b, c) with $1 \le a \le b \le c$ that satisfy $a + b + c = 77$ and:

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{5}.$$

What is the value of the sum $\sum_{a,b,c \in S} a \cdot b \cdot c$?

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11 Given a rational number r that, when expressed in base-10, is a repeating, non-terminating decimal, we define $f(r)$ to be the number of digits in the decimal representation of r that are after the decimal point but before the repeating part of r. For example, $f(1.2\overline{7}) = 0$ and $f(0.35\overline{2}) = 2$. What is the smallest positive integer n such that $\frac{1}{n}$, $\frac{2}{n}$, and $\frac{4}{n}$ are non-terminating decimals, where $f\left(\frac{1}{n}\right) = 3$, $f\left(\frac{2}{n}\right) = 3$, and $f\left(\frac{4}{n}\right) = 2$?.

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12 Alice is stacking balls on the ground in three layers using two sizes of balls: small and large. All small balls are the same size, as are all large balls. For the first layer, she uses 6 identical large balls A, B, C, D, E, and F all touching the ground and so that D, E, F touch each other, A touches E and F, B touches D and F, and C touches D and E. For the second layer, she uses 3 identical small balls, G, H, and I; G touches A, E, and F, H touches B, D, and F, and I touches C, D, and E. Obviously, the small balls do not intersect the ground. Finally, for the top layer, she uses one large ball that touches D, E, F, G, H, and I. If the large balls have volume 2015, the sum of the volumes of all the balls in the pyramid can be written in the form $a\sqrt{b}+c$ for integers a, b, c where no integer square larger than 1 divides b. What is $a+b+c$? (This diagram may not have the correct scaling, but just serves to clarify the layout of the problem.)

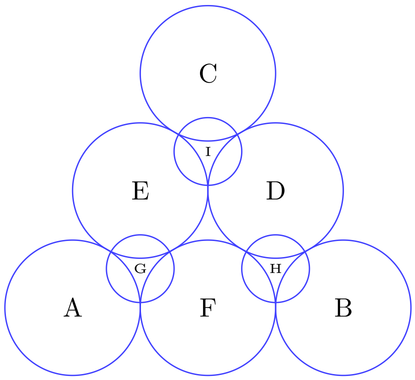

Figure 1: The projection of the balls onto the ground

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13 We define $\lfloor x \rfloor$ as the largest integer less than or equal to x. What is

$$\left\lfloor \frac{5^{2017015}}{5^{2015}+7} \right\rfloor \mod 1000?$$

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14 Marie is painting a 4×4 grid of identical square windows. Initially, they are all orange but she wants to paint 4 of them black. How many ways can she do this up to rotation and reflection?

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15 Let S be the set of ordered integer pairs (x, y) such that $0 < x < y < 42$ and there exists some integer n such that $x^6 - y^6 \mid n^2 + 2015^2$. What is the sum $\sum_{(x_i,y_i)\in S} x_i y_i$?

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16 Let p, u, m, a, c be positive real numbers satisfying $5p^5 + 4u^5 + 3m^5 + 2a^5 + c^5 = 91$. What is the maximum possible value of:

$$18pumac + 2(2+p)^2 + 23(1+ua)^2 + 15(3+mc)^2?$$
