HMMT Invitational Competition 2013

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1 Let S be a set of size n, and k be a positive integer. For each $1 \leq i \leq kn$, there is a subset $S_i \subset S$ such that $|S_i| = 2$. Furthermore, for each $e \in S$, there are exactly $2k$ values of i such that $e \in S_i$. Show that it is possible to choose one element from S_i for each $1 \leq i \leq kn$ such that every element of S is chosen exactly k times.

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2 Find all functions $f : R \to R$ such that, for all real numbers x, y,

$$(x - y)(f(x) - f(y)) = f(x - f(y))f(f(x) - y).$$

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3 Triangle ABC is inscribed in a circle ω such that $\angle A = 60^o$ and $\angle B = 75^o$. Let the bisector of angle A meet BC and ω at E and D, respectively. Let the reflections of A across D and C be D' and C' , respectively. If the tangent to ω at A meets line BC at P, and the circumcircle of APD' meets line AC at $F \neq A$, prove that the circumcircle of $C'FE$ is tangent to BC at E.

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4 A subset $U \subset R$ is open if for any $x \in U$, there exist real numbers a, b such that $x \in (a, b) \subset U$. Suppose $S \subset R$ has the property that any open set intersecting $(0, 1)$ also intersects S. Let T be a countable collection of open sets containing S. Prove that the intersection of all of the sets of T is not a countable subset of R.
(A set Γ is countable if there exists a bijective function $f : \Gamma \to Z$.)

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5 I'd really appreciate help on this.

(a) Given a set X of points in the plane, let $f_X(n)$ be the largest possible area of a polygon with at most n vertices, all of which are points of X. Prove that if m, n are integers with $m \geq n > 2$ then $f_X(m) + f_X(n) \geq f_X(m + 1) + f_X(n - 1)$.

(b) Let P_0 be a 1×2 rectangle (including its interior) and inductively define the polygon P_i to be the result of folding P_{i-1} over some line that cuts P_{i-1} into two connected parts. The diameter of a polygon P_i is the maximum distance between two points of P_i. Determine the smallest possible diameter of P_{2013}.
