Princeton University Math Competition 2012

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by parmenides51

– Geometry

A1 / B4 Three circles, with radii of $1, 1$, and 2, are externally tangent to each other. The minimum possible area of a quadrilateral that contains and is tangent to all three circles can be written as $a + b\sqrt{c}$ where c is not divisible by any perfect square larger than 1. Find $a + b + c$

A2 / B5 Two circles centered at O and P have radii of length 5 and 6 respectively. Circle O passes through point P. Let the intersection points of circles O and P be M and N. The area of triangle $\triangle MNP$ can be written in simplest form as a/b. Find $a + b$.

A3 Six ants are placed on the vertices of a regular hexagon with an area of 12. At each point in time, each ant looks at the next ant in the hexagon (in counterclockwise order), and measures the distance, s, to the next ant. Each ant then proceeds towards the next ant at a speed of $\frac{s}{100}$ units per year. After T years, the ants' new positions are the vertices of a new hexagon with an area of 4. T is of the form $a \ln b$, where b is square-free. Find $a + b$.

A4 / B6 A square is inscribed in an ellipse such that two sides of the square respectively pass through the two foci of the ellipse. The square has a side length of 4. The square of the length of the minor axis of the ellipse can be written in the form $a + b\sqrt{c}$ where a, b, and c are integers, and c is not divisible by the square of any prime. Find the sum $a + b + c$.

A5 Let $\triangle ABC$ be a triangle with $\angle BAC = 45^o$, $\angle BCA = 30^o$, and $AB = 1$. Point D lies on segment AC such that $AB = BD$. Find the square of the length of the common external tangent to the circumcircles of triangles $\triangle BDC$ and $\triangle ABC$.

A6 Consider a pool table with the shape of an equilateral triangle. A ball of negligible size is initially placed at the center of the table. After it has been hit, it will keep moving in the direction it was hit towards and bounce off any edges with perfect symmetry. If it eventually reaches the midpoint of any edge, we mark the midpoint of the entire route that the ball has travelled through. Repeating this experiment, how many points can we mark at most?
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png

A7 An octahedron (a solid with 8 triangular faces) has a volume of 1040. Two of the spatial diagonals intersect, and their plane of intersection contains four edges that form a cyclic quadrilateral. The third spatial diagonal is perpendicularly bisected by this plane and intersects the plane at the circumcenter of the cyclic quadrilateral. Given that the side lengths of the cyclic

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quadrilateral are $7, 15, 24, 20$, in counterclockwise order, the sum of the edge lengths of the entire octahedron can be written in simplest form as a/b. Find $a + b$.

A8 Cyclic quadrilateral $ABCD$ has side lengths $AB = 2, BC = 3, CD = 5, AD = 4$.
Find $\sin A \sin B(\cot A/2 + \cot B/2 + \cot C/2 + \cot D/2)^2$.
Your answer can be written in simplest form as a/b. Find $a + b$.

B1 During chemistry labs, we oftentimes fold a disk-shaped filter paper twice, and then open up a flap of the quartercircle to form a cone shape, as in the diagram. What is the angle θ, in degrees, of the bottom of the cone when we look at it from the side?
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B2 A 6-inch-wide rectangle is rotated 90 degrees about one of its corners, sweeping out an area of 45π square inches, excluding the area enclosed by the rectangle in its starting position. Find the rectangle's length in inches.

B3 Let A be a regular 12-sided polygon. A new 12-gon B is constructed by connecting the midpoints of the sides of A. The ratio of the area of B to the area of A can be written in simplest form as $(a + \sqrt{b})/c$, where a, b, c are integers. Find $a + b + c$.

B7 Assume the earth is a perfect sphere with a circumference of 60 units. A great circle is a circle on a sphere whose center is also the center of the sphere. There are three train tracks on three great circles of the earth. One is along the equator and the other two pass through the poles, intersecting at a 90 degree angle. If each track has a train of length L traveling at the same speed, what is the maximum value of L such that the trains can travel without crashing?

B8 A cyclic quadrilateral $ABCD$ has side lengths $AB = 3, BC = AD = 5$, and $CD = 8$. The radius of its circumcircle can be written in the form $a\sqrt{b}/c$, where a, b, c are positive integers, a, c are relatively prime, and b is not divisible by the square of any prime. Find $a + b + c$.

– Combinatorics

A1 / B2 If the probability that the sum of three distinct integers between 16 and 30 (inclusive) is even can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m + n$.

A2 / B3 How many ways are there to arrange the 6 permutations of the tuple $(1, 2, 3)$ in a sequence, such that each pair of adjacent permutations contains at least one entry in common?
For example, a valid such sequence is given by $(3, 2, 1) - (2, 3, 1) - (2, 1, 3) - (1, 2, 3) - (1, 3, 2) - (3, 1, 2)$.

A3 / B5 Jim has two fair 6-sided dice, one whose faces are labelled from 1 to 6, and the second whose faces are labelled from 3 to 8. Twice, he randomly picks one of the dice (each die equally likely) and rolls it. Given the sum of the resulting two rolls is 9, if $\frac{m}{n}$ is the probability he rolled the same die twice where m, n are relatively prime positive integers, then what is $m + n$?

A4 / B6 How many (possibly empty) sets of lattice points $\{P_1, P_2, ..., P_M\}$, where each point $P_i = (x_i, y_i)$ for $x_i, y_i \in \{0, 1, 2, 3, 4, 5, 6\}$, satisfy that the slope of the line P_iP_j is positive for each $1 \le i < j \le M$? An infinite slope, e.g. P_i is vertically above P_j , does not count as positive.

A5 / B7 5 people stand in a line facing one direction. In every round, the person at the front moves randomly to any position in the line, including the front or the end. Suppose that $\frac{m}{n}$ is the expected number of rounds needed for the last person of the initial line to appear at the front of the line, where m and n are relatively prime positive integers. What is $m + n$?

A6 Two white pentagonal pyramids, with side lengths all the same, are glued to each other at their regular pentagon bases. Some of the resulting 10 faces are colored black. How many rotationally distinguishable colorings may result?

A7 / B8 A PUMaC grader is grading the submissions of forty students $s_1, s_2, ..., s_{40}$ for the individual finals round, which has three problems. After grading a problem of student s_i, the grader either: • grades another problem of the same student, or • grades the same problem of the student s_{i-1} or s_{i+1} (if $i > 1$ and $i < 40$, respectively).

He grades each problem exactly once, starting with the first problem of s_1 and ending with the third problem of s_{40}. Let N be the number of different orders the grader may grade the students' problems in this way. Find the remainder when N is divided by 100.

A8 Proctors Andy and Kristin have a PUMaC team of eight students labelled $s_1, s_2, ..., s_8$ (the PUMaC staff being awful with names). The following occurs: 1. Andy tells the students to arrange themselves in a line in arbitrary order. 2. Kristin tells each student s_i to move to the current spot of student s_j , where $j \equiv 3i + 1 \pmod 8$. 3. Andy tells each student s_i to move to the current spot of the student who was in the ith position of the line after step 1.

How many possible orders can the students be in now?

B1 Your friend sitting to your left (or right?) is unable to solve any of the eight problems on his or her Combinatorics B test, and decides to guess random answers to each of them. To your astonishment, your friend manages to get two of the answers correct. Assuming your friend has equal probability of guessing each of the questions correctly, what is the average possible value of your friend's score? Recall that each question is worth the point value shown at the beginning of each question.

B4 For a set S of integers, define $\max(S)$ to be the maximal element of S. How many non-empty subsets $S \subseteq \{1, 2, 3, ..., 10\}$ satisfy $\max(S) \le |S| + 2$?

– Algebra

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A1 Compute the smallest positive integer a for which

$$\sqrt{a+\sqrt{a+...}}-\frac{1}{a+\frac{1}{a+...}}>7$$

A2 / B4 If x, y, and z are real numbers with $\frac{x-y}{z}+\frac{y-z}{x}+\frac{z-x}{y}=36$, find $2012+\frac{x-y}{z}\cdot\frac{y-z}{x}\cdot\frac{z-x}{y}$.

A3 / B6 Compute $\Sigma_{n=1}^{\infty}\frac{n+1}{n^2(n+2)^2}$.
Your answer in simplest form can be written as a/b, where a, b are relatively-prime positive integers. Find $a+b$.

A4 / B7 Let f be a polynomial of degree 3 with integer coefficients such that $f(0)=3$ and $f(1)=11$. If f has exactly 2 integer roots, how many such polynomials f exist?

A5 What is the smallest natural number n greater than 2012 such that the polynomial $f(x)=(x^6+x^4)^n-x^{4n}-x^6$ is divisible by $g(x)=x^4+x^2+1$?

A6 Let an be a sequence such that $a_0=0$ and: $a_{3n+1}=a_{3n}+1=a_n+1$ $a_{3n+2}=a_{3n}+2=a_n+2$ for all natural numbers n. How many n less than 2012 have the property that $a_n=7$?

A7 / B8 Let a_n be a sequence such that $a_1=1$ and $a_{n+1}=\lfloor a_n+\sqrt{a_n}+\frac{1}{2}\rfloor$, where $\lfloor x\rfloor$ denotes the greatest integer less than or equal to x. What are the last four digits of a_{2012}?

A8 If n is an integer such that $n\geq 2^k$ and $n<2^{k+1}$, where $k=1000$, compute the following:

$$n-\left(\lfloor\frac{n-2^0}{2^1}\rfloor+\lfloor\frac{n-2^1}{2^2}\rfloor+...+\lfloor\frac{n-2^{k-1}}{2^k}\rfloor\right)$$

B1 Find the largest n such that the last nonzero digit of $n!$ is 1.

B2 Define a sequence a_n such that $a_n=a_{n-1}-a_{n-2}$. Let $a_1=6$ and $a_2=5$. Find $\Sigma_{n=1}^{1000}a_n$.

B3 Evaluate $\sqrt[3]{26+15\sqrt{3}}+\sqrt[3]{26-15\sqrt{3}}$

B5 Considering all numbers of the form $n=\lfloor\frac{k^3}{2012}\rfloor$, where $\lfloor x\rfloor$ denotes the greatest integer less than or equal to x, and k ranges from 1 to 2012, how many of these n's are distinct?

– Number Theory

A1 / B4 Albert has a very large bag of candies and he wants to share all of it with his friends. At first, he splits the candies evenly amongst his 20 friends and himself and he finds that there are five left over. Ante arrives, and they redistribute the candies evenly again. This time, there are three left over. If the bag contains over 500 candies, what is the fewest number of candies the bag can contain?

A2 / B5 How many ways can 2^{2012} be expressed as the sum of four (not necessarily distinct) positive squares?

A3 Let the sequence $\{x_n\}$ be defined by $x_1 \in \{5, 7\}$ and, for $k \ge 1, x_{k+1} \in \{5^{x_k}, 7^{x_k}\}$. For example, the possible values of x_3 are $5^{5^5}, 5^{5^7}, 5^{7^5}, 5^{7^7}, 7^{5^5}, 7^{5^7}, 7^{7^5}$, and 7^{7^7}. Determine the sum of all possible values for the last two digits of x_{2012}.

A4 / B7 Find the sum of all possible sums $a + b$ where a and b are nonnegative integers such that $4^a + 2^b + 5$ is a perfect square.

A5 Call a positive integer x a leader if there exists a positive integer n such that the decimal representation of x^n starts (<u>not ends</u>) with 2012. For example, 586 is a leader since $586^3 = 201230056$. How many leaders are there in the set $\{1, 2, 3, ..., 2012\}$?

A6 Let $p_1 = 2012$ and $p_n = 2012^{p_{n-1}}$ for $n > 1$. Find the largest integer k such that $p_{2012} - p_{2011}$ is divisible by 2011^k.

A7 Let a, b, and c be positive integers satisfying $a^4 + a^2b^2 + b^4 = 9633\ 2a^2 + a^2b^2 + 2b^2 + c^5 = 3605$. What is the sum of all distinct values of $a + b + c$?

A8 Find the largest possible sum $m + n$ for positive integers $m, n \le 100$ such that $m + 1 \equiv 3$ (mod 4) and there exists a prime number p and nonnegative integer a such $\frac{m^{2n-1}-1}{m-1} = m^n + p^a$
.

B1 When some number a^2 is written in base b, the result is 144_b. a and b also happen to be integer side lengths of a right triangle. If a and b are both less than 20, find the sum of all possible values of a.

B2 Let M be the smallest positive multiple of 2012 that has 2012 divisors.
Suppose M can be written as $\Pi_{k=1}^{n} p_k^{a_k}$ where the p_k's are distinct primes and the a_k's are positive integers.
Find $\Sigma_{k=1}^{n}(p_k + a_k)$

B3 How many factors of $(20^{12})^2$ less than 20^{12} are not factors of 20^{12} ?

B6 Let $f_n(x) = n+x^2$. Evaluate the product $gcd\{f_{2001}(2002), f_{2001}(2003)\} \times gcd\{f_{2011}(2012), f_{2011}(2013)\} \times gcd\{f_{2021}(2022), f_{2021}(2023)\}$, where $gcd\{x, y\}$ is the greatest common divisor of x and y

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– Individual Finals

A1 Let p be a prime number greater than 5. Prove that there exists a positive integer n such that p divides $20^n + 15^n - 12^n$.

A2 Let a, b, c be real numbers such that $a + b + c = abc$. Prove that $\frac{1}{a^2+1} + \frac{1}{b^2+1} + \frac{1}{c^2+1} \geq \frac{3}{4}$.

A3 Let ABC be a triangle with incenter I, and let D be the foot of the angle bisector from A to BC. Let Γ be the circumcircle of triangle BIC, and let PQ be a chord of Γ passing through D. Prove that AD bisects $\angle PAQ$.

B1 Let q be a fixed odd prime. A prime p is said to be *orange* if for every integer a there exists an integer r such that $r^q \equiv a \pmod{p}$. Prove that there are infinitely many *orange* primes.

B2 Let $O_1, O_2, ..., O_{2012}$ be 2012 circles in the plane such that no circle intersects or contains anyother circle and no two circles have the same radius. For each $1 \leq i < j \leq 2012$, let $P_{i,j}$ denotethe point of intersection of the two external tangent lines to O_i and O_j, and let T be the set of all $P_{i,j}$ (so $|T| = \binom{2012}{2} = 2023066$). Suppose there exists a subset $S \subset T$ with $|S| = 2021056$ such that all points in S lie on the same line. Prove that all points in T lie on the same line.

B3 Find, with proof, all pairs (x, y) of integers satisfying the equation $3x^2 + 4 = 2y^3$.

Team Round Time limit: 20 minutes.

Fill in the crossword above with answers to the problems below.

Notice that there are three directions instead of two. You are probably used to ”down” and ”across,” but this crossword has ”1,” $e^{4\pi i/3}$, and $e^{5\pi i/3}$. You can think of these labels as complex numbers pointing in the direction to fill in the spaces. In other words ”1” means ”across”, $e^{4\pi i/3}$ means ”down and to the left,” and $e^{5\pi i/3}$ means ”down and to the right.”

To fill in the answer to, for example, 12 across, start at the hexagon labeled 12, and write the digits, proceeding to the right along the gray line. (Note: 12 across has space for exactly 5 digits.)

Each hexagon is worth one point, and must be filled by something from the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Note that π is not in the set, and neither is i, nor $\sqrt{2}$, nor ♡,etc.

None of the answers will begin with a 0.

”Concatenate a and b” means to write the digits of a, followed by the digits of b. For example, concatenating 10 and 3 gives 103. (It’s not the same as concatenating 3 and 10.)

Calculators are allowed!

THIS SHEET IS PROVIDED FOR YOUR REFERENCE ONLY. DO NOT TURN IN THIS SHEET. TURN IN THE OFFICIAL ANSWER SHEET PROVIDED TO THE TEAM. OTHERWISE YOU WILL GET A SCORE OF ZERO! ZERO! ZERO! AND WHILE SOMETIMES "!" MEANS FACTORIAL, IN THIS CASE IT DOES NOT.

Good luck, and have fun!
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Across (1)

A 3. (3 digits) Suppose you draw 5 vertices of a convex pentagon (but not the sides!). Let N be the number of ways you can draw at least 0 straight line segments between the vertices so that no two line segments intersect in the interior of the pentagon. What is $N - 64$? (Note what the question is asking for! You have been warned!)

A 5. (3 digits) Among integers $\{1, 2, ..., 10^{2012}\}$, let n be the number of numbers for which the sum of the digits is divisible by 5. What are the first three digits (from the left) of n?

A 6. (3 digits) Bob is punished by his math teacher and has to write all perfect squares, one after another. His teacher's blackboard has space for exactly 2012 digits. He can stop when he cannot fit the next perfect square on the board. (At the end, there might be some space left on the board - he does not write only part of the next perfect square.) If n^2 is the largest perfect square he writes, what is n?

A 8. (3 digits) How many positive integers n are there such that $n \le 2012$, and the greatest common divisor of n and 2012 is a prime number?

A 9. (4 digits) I have a random number machine generator that is very good at generating integers between 1 and 256, inclusive, with equal probability. However, right now, I want to produce a random number between 1 and n, inclusive, so I do the following: • I use my machine to generate a number between 1 and 256. Call this a. • I take a and divide it by n to get remainder r. If $r \ne 0$, then I record r as the randomly generated number. If $r = 0$, then I record n instead.
Note that this process does not necessarily produce all numbers with equal probability, but that is okay. I apply this process twice to generate two numbers randomly between 1 and 10. Let p be the probability that the two numbers are equal. What is $p \cdot 2^{16}$?

A 12. (5 digits) You and your friend play the following dangerous game. You two start off at some point (x, y) on the plane, where x and y are nonnegative integers.
When it is player A's turn, A tells his opponent B to move to another point on the plane. Then A waits for a while. If B is not eaten by a tiger, then A moves to that point as well.
From a point (x, y) there are three places A can tell B to walk to: leftwards to $(x - 1, y)$, downwards to $(x, y - 1)$, and simultaneously downwards and leftwards to $(x - 1, y - 1)$. However, you cannot move to a point with a negative coordinate.
Now, what was this about being eaten by a tiger? There is a tiger at the origin, which will eat the

first person that goes there! Needless to say, you lose if you are eaten.
Consider all possible starting points (x, y) with $0 \leq x \leq 346$ and $0 \leq y \leq 346$, and x and y are not both zero. Also suppose that you two play strategically, and you go first (i.e., by telling your friend where to go). For how many of the starting points do you win?

Down and to the left $e^{4\pi i/3}$

DL 2. (2 digits) ABCDE is a pentagon with $AB = BC = CD = \sqrt{2}$, $\angle ABC = \angle BCD = 120$ degrees, and $\angle BAE = \angle CDE = 105$ degrees. Find the area of triangle $\triangle BDE$. Your answer in its simplest form can be written as $\frac{a+\sqrt{b}}{c}$, where where a, b, c are integers and b is square-free. Find abc.

DL 3. (3 digits) Suppose x and y are integers which satisfy

$$\frac{4x^2}{y^2} + \frac{25y^2}{x^2} = \frac{10055}{x^2} + \frac{4022}{y^2} + \frac{2012}{x^2y^2} - 20.$$

What is the maximum possible value of $xy - 1$?

DL 5. (3 digits) Find the area of the set of all points in the plane such that there exists a square centered around the point and having the following properties: • The square has side length $7\sqrt{2}$. • The boundary of the square intersects the graph of $xy = 0$ at at least 3 points.

DL 8. (3 digits) Princeton Tiger has a mom that likes yelling out math problems. One day, the following exchange between Princeton and his mom occurred: • Mom: Tell me the number of zeros at the end of $2012!$ • PT: Huh? 2012 ends in 2, so there aren't any zeros. • Mom: No, the exclamation point at the end was not to signify me yelling. I was not asking about 2012, I was asking about $2012!$.
What is the correct answer?

DL 9. (4 digits) Define the following: • $A = \sum_{n=1}^{\infty} \frac{1}{n^6}$ • $B = \sum_{n=1}^{\infty} \frac{1}{n^6+1}$ • $C = \sum_{n=1}^{\infty} \frac{1}{(n+1)^6}$ • $D = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^6}$ • $E = \sum_{n=1}^{\infty} \frac{1}{(2n+1)^6}$
Consider the ratios $\frac{B}{A}, \frac{C}{A}, \frac{D}{A}, \frac{E}{A}$. Exactly one of the four is a rational number. Let that number be r/s, where r and s are nonnegative integers and $gcd\,(r, s) = 1$. Concatenate r, s.
(It might be helpful to know that $A = \frac{\pi^6}{945}$.)

DL 10. (3 digits) You have a sheet of paper, which you lay on the xy plane so that its vertices are at $(-1, 0)$, $(1, 0)$, $(1, 100)$, $(-1, 100)$. You remove a section of the bottom of the paper by cutting along the function $y = f(x)$, where f satisfies $f(1) = f(-1) = 0$. (In other words, you keep the bottom two vertices.)
You do this again with another sheet of paper. Then you roll both of them into identical cylinders, and you realize that you can attach them to form an L-shaped elbow tube.
We can write $f\left(\frac{1}{3}\right) + f\left(\frac{1}{6}\right) = \frac{a+\sqrt{b}}{\pi c}$, where a, b, c are integers and b is square-free. $Find a + b + c$.

DL 11. (3 digits) Let

$$\Xi(x) = 2012(x-2)^2 + 278(x-2)\sqrt{2012 + e^{x^2-4x+4}} + 1392 + (x^2-4x+4)e^{x^2-4x+4}$$

find the area of the region in the xy-plane satisfying:

$$\{x \geq 0 \ \ and x \leq 4 \ \ and \ \ y \geq 0 \ \ and \ \ y \leq \sqrt{\Xi(x)}\}$$

DL 13. (3 digits) Three cones have bases on the same plane, externally tangent to each other. The cones all face the same direction. Two of the cones have radii of 2, and the other cone has a radius of 3. The two cones with radii 2 have height 4, and the other cone has height 6. Let V be the volume of the tetrahedron with three of its vertices as the three vertices of the cones and the fourth vertex as the center of the base of the cone with height 6. Find V^2.

Down and to the right $e^{5\pi i/3}$

DR 1. (2 digits) For some reason, people in math problems like to paint houses. Alice can paint a house in one hour. Bob can paint a house in six hours. If they work together, it takes them seven hours to paint a house. You might be thinking "What? That's not right!" but I did not make a mistake.
When Alice and Bob work together, they get distracted very easily and simultaneously send text messages to each other. When they are texting, they are not getting any work done.
When they are not texting, they are painting at their normal speeds (as if they were working alone). Carl, the owner of the house decides to check up on their work. He randomly picks a time during the seven hours. The probability that they are texting during that time can be written as r/s, where r and s are integers and $gcd\,(r,s)=1$. What is $r+s$?

DR 4. (3 digits) Let $a_1 = 2+\sqrt{2}$ and $b_1 = \sqrt{2}$, and for $n \geq 1$, $a_{n+1} = |a_n - b_n|$ and $b_{n+1} = a_n + b_n$. The minimum value of $\frac{a_n^2+a_nb_n-6b_n^2}{6b_n^2-a_n^2}$ can be written in the form $a\sqrt{b}-c$, where a,b,c are integers and b is square-free. Concatenate c,b,a (in that order!).

DR 7. (3 digits) How many solutions are there to $a^{503}+b^{1006}=c^{2012}$, where a,b,c are integers and $|a|$,$|b|$, $|c|$ are all less than 2012?

PS. You should use hide for answers.

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