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by ThE-dArK-lOrD, Kayak

A1 Consider the sequence of positive integers defined by $s_1, s_2, s_3, \ldots$ of positive integers defined by

-$s_1 = 2$, and
-for each positive integer n, s_{n+1} is equal to s_n plus the product of prime factors of s_n.

The first terms of the sequence are $2, 4, 6, 12, 18, 24$.
Prove that the product of the 2019 smallest primes is a term of the sequence.

A2 Consider the operation $*$ that takes pair of integers and returns an integer according to the rule

$$a * b = a \times (b+1).$$

-For each positive integer n, determine all permutations $a_1, a_2, \ldots, a_n$ of the set $\{1, 2, \ldots, n\}$ that maximise the value of

$$(\cdots((a_1 * a_2) * a_3) * \cdots * a_{n-1}) * a_n.$$

-For each positive integer n, determine all permutations $b_1, b_2, \ldots, b_n$ of the set $\{1, 2, \ldots, n\}$ that maximise the value of

$$b_1 * (b_2 * (b_3 * \cdots * (b_{n-1} * b_n) \cdots)).$$

A3 For some positive integer n, a coin will be ipped n times to obtain a sequence of$n$ heads and tails. For each ip of the coin, there is probability p of obtaining a head and probability $1-p$ of obtaining a tail, where $0<p<1$ is a rational number.
Kim writes all 2^n possible sequences of$n$ heads and tails in two columns, with some sequences in the left column and the remaining sequences in the right column. Kim would like the sequence produced by the coin ips to appear in the left column with probability $1/2$.
Determine all pairs (n,p) for which this is possible.

A4 Suppose $x_1, x_2, x_3, \ldots$ is a strictly decreasing sequence of positive real numbers such that the series $x_1 + x_2 + x_3 + \cdots$ diverges.

Is it necessary true that the series $\sum_{n=2}^{\infty} \min\left\{x_n, \frac{1}{n\log(n)}\right\}$ diverges?

B1 Determine all pairs (a, b) of real numbers with $a \leqslant b$ that maximise the integral

$$\int_a^b e^{\cos(x)}(380 - x - x^2)\mathrm{d}x.$$

B2 For each odd prime number p, prove that the integer

$$1! + 2! + 3! + \cdots + p! - \left\lfloor \frac{(p-1)!}{e} \right\rfloor$$

is divisible by p
(Here, e denotes the base of the natural logarithm and $\lfloor x \rfloor$ denotes the largest integer that is less than or equal to x.)

B3 Let G be a nite simple graph and let k be the largest number of vertices of any clique in G. Suppose that we label each vertex of$G$ with a non-negative real number, so that the sum of all such labels is 1. Dene the*value of an edge* to be the product of the labels of the two vertices at its ends. Dene the *value of a labelling* to be the sum of values of the edges.
Prove that the maximum possible value of a labelling of$G$ is $\frac{k-1}{2k}$.
(A *nite simple graph*is a graph with nitely many vertices, in which each edge connects two distinct vertices and no two edges connect the same two vertices. A *clique* in a graph is a set of vertices in which any two are connected by an edge.)

B4 A *binary string*is a sequence, each of whose terms is 0 or 1. A set $\mathcal{B}$ of binary strings is dened inductively according to the following rules.

-The binary string 1 is in $\mathcal{B}$.
-If $s_1, s_2, \ldots, s_n$ is in $\mathcal{B}$ with n odd, then both $s_1, s_2, \ldots, s_n, 0$ and $0, s_1, s_2, \ldots, s_n$ are in $\mathcal{B}$.
-If $s_1, s_2, \ldots, s_n$ is in $\mathcal{B}$ with n even, then both $s_1, s_2, \ldots, s_n, 1$ and $1, s_1, s_2, \ldots, s_n$ are in $\mathcal{B}$.
-No other binary strings are in $\mathcal{B}$.

For each positive integer n, let b_n be the number of binary strings in $\mathcal{B}$ of length n.

-Prove that there exist constants $c_1, c_2 > 0$ and $1.6 < \lambda_1, \lambda_2 < 1.9$ such that $c_1\lambda_1^n < b_n < c_2\lambda_2^n$ for all positive integer n.
-Determine $\liminf_{n\to\infty} \sqrt[n]{b_n}$ and $\limsup_{n\to\infty} \sqrt[n]{b_n}$

Note: The problem is open in the sense that no solution is currently known to part (b).
