www.artofproblemsolving.com/community/c986174

by BigSams, Pinko, billzhao, ductiena1k43, abhinavzandubalm, k.vasilev, MarizOzawa

–	First Round
1	We define the sequence $a_n = (2n)^2 + 1$ for each natural number n. We will call one number *bad*, if there dont exist natural numbers $a > 1$ and $b > 1$ such that $a_n = a^2 + b^2$. Prove that the natural number n is *bad*, if and only if a_n is prime.
2	There are some boys and girls that study in a school. A group of boys is called *sociable*, if each girl knows at least one of the boys in the group. A group of girls is called *sociable*, if each boy knows at least one of the girls in the group. If the number of *sociable* groups of boys is odd, prove that the number of *sociable* groups of girls is also odd.
3	The perpendicular bisector of AB of an acute ΔABC intersects BC and the continuation of AC in points P and Q respectively. M and N are the middle points of side AB and segment PQ respectively. If the lines AB and CN intersect in point D, prove that ΔABC and ΔDCM have a common orthocenter.
4	The diagonals AC and BD of a convex quadrilateral $ABCD$ intersect in point M. The angle bisector of $\angle ACD$ intersects the ray $\overrightarrow{BA}$ in point K. If $MA.MC + MA.CD = MB.MD$, prove that $\angle BKC = \angle CDB$.
5	Prove that there exist a natural number a, for which 999 divides $2^{5n} + a.5^n$ for $\forall$ odd $n \in \mathbb{N}$ and find the smallest such a.
6	Find all odd numbers $n \in \mathbb{N}$, for which the number of all natural numbers, that are no bigger than n and coprime with n, divides $n^2 + 3$.
7	Let a, b, c be positive real numbers such that $abc = 8$. Prove that $$\frac{a^2}{\sqrt{(1+a^3)(1+b^3)}} + \frac{b^2}{\sqrt{(1+b^3)(1+c^3)}} + \frac{c^2}{\sqrt{(1+c^3)(1+a^3)}} \geq \frac{4}{3}$$
8	Find whether the number of powers of 2, which have a digit sum smaller than 2019^{2019}, is finite or infinite.
–	Second Round

1 A football tournament is played between 5 teams, each two of which playing exactly one match. 5 points are awarded for a victory and 0 for a loss. In case of a draw 1 point is awarded to both teams, if no goals are scored, and 2 if they have scored any. In the final ranking the five teams had points that were 5 consecutive numbers. Determine the least number of goals that could be scored in the tournament.

---

2 In ΔABC with $\angle ACB = 135^\circ$, are chosen points M and N on side AB, so that $\angle MCN = 90^\circ$. Segments MD and NQ are angle bisectors of ΔAMC and ΔNBC respectively. Prove that the reflection of C in line PQ lies on the line AB.

---

3 There are 365 cards with 365 different numbers. Each step, we can choose 3 cards a_i, a_j, a_k and we know the order of them (examble: $a_i < a_j < a_k$). With 2000 steps, can we order 365 cards from smallest to biggest??

---

4 For a quadrilateral $ABCD$ is given that $\angle CBD = 2\angle ADB$, $\angle ABD = 2\angle CDB$, and $AB = CB$. Prove that $AD = CD$.

---

5 Let A be the number of 2019-digit numbers, that is made of 2 different digits (For example $10\underbrace{1...1}_{2016}0$ is such number). Determine the highest power of 3 that divides A.

---

6 There are n kids. From each two at least one of them has sent an SMS to the other. For each kid A, among the kids on which A has sent an SMS, exactly 10

---

7 Let G be a bipartite graph in which the greatest degree of a vertex is 2019. Let m be the least natural number for which we can color the edges of G in m colors so that each two edges with a common vertex from G are in different colors. Show that m doesnt depend on G and find its value.

---

8 Solve the following equation in integers: $4n^4 + 7n^2 + 3n + 6 = m^3$.

– Third Round

---

1 Let p_1, p_2, p_3, and p be prime numbers. Prove that there exist $x, y \in \mathbb{Z}$ such that $y^2 \equiv p_1x^4 - p_1p_2^2p_3^2 \,(mod\, p)$.

---

2 ΔABC is a triangle with center I of its inscribed circle and B_1 and C_1 are feet of its angle bisectors through B and C. Let S be the middle point on the arc $\widehat{BAC}$ of the circumscribed circle of ΔABC (denoted with Ω) and let ω_a be the excircle of ΔABC opposite to A. Let $\omega_a(I_a)$ be tangent to AB and AC in points D and E respectively and $SI \cap \Omega = \{S, P\}$. Let M be the middle point of DE and N be the middle point of SI. If $MN \cap AP = K$, prove that $KI_a \perp B_1C_1$.

---

3 We are given a non-obtuse ΔABC $(BC > AC)$ with an altitude CD $(D \in AB)$, center O of its circumscribed circle, and a middle point M of its side AB. Point E lies on the ray $\overrightarrow{BA}$ in such way that $AE.BE = DE.ME$. If the line OE bisects the area of ΔABC and $CO = CD.cos\angle ACB$, determine the angles of ΔABC.

---

4 On a competition called *"Mathematical duels"* students were given n problems and each student solved exactly 3 of them. For each two students there is at most one problem that is solved from both of them. Prove that, if $s \in \mathbb{N}$ is a number for which $s^2 - s + 1 < 2n$, then there are s problems among the n, no three of which solved by one student.

---

5 Let $a > 0$ and $12a + 5b + 2c > 0$. Prove that it is impossible for the equation $ax^2 + bx + c = 0$ to have two real roots in the interval $(2, 3)$.

---

6 Does there exist a function $f : \mathbb{N} \to \mathbb{N}$ such that for all integers $n \geq 2$,

$$f(f(n-1)) = f(n+1) - f(n)\,?$$

---

7 The function $f : \mathbb{R} \to \mathbb{R}$ is such that $f(x + 1) = 2f(x)$ for $\forall\ x \in \mathbb{R}$ and $f(x) = x(x - 1)$ for $\forall\ x \in (0, 1]$. Find the greatest real number m, for which the inequality $f(x) \geq -\frac{8}{9}$ is true for $\forall$ $x \in (-\infty, m]$.

---

8 Find all polynomials $f \in Z[X]$, such that for each odd prime p

$$f(p)|(p-3)! + \frac{p+1}{2}.$$

– Fourth Round

---

1 The points M and N are on the side BC of ΔABC, so that $BM = CN$ and M is between B and N. Points $P \in AN$ and $Q \in AM$ are such that $\angle PMC = \angle MAB$ and $\angle QNB = \angle NAC$. Prove that $\angle QBC = \angle PCB$.

---

2 Let n be a natural number. At first the cells of a table $2n$ x $2n$ are colored in white. Two players A and B play the following game. First is A who has to color m arbitrary cells in red and after that B chooses n rows and n columns and color their cells in black. Player A wins, if there is at least one red cell on the board. Find the least value of m for which A wins no matter how B plays.

---

3 The natural number $n > 1$ is such that there exist $a \in \mathbb{N}$ and a prime number q which satisfy the following conditions:
1) q divides $n - 1$ and $q > \sqrt{n} - 1$

2) n divides $a^{n-1} - 1$
3) $gcd(a^{\frac{n-1}{q}} - 1, n) = 1$.
Is it possible for n to be a composite number?

---

4 Is it true that for $\forall$ prime number p, there exist non-constant polynomials P and Q with $P, Q \in \mathbb{Z}[x]$ for which the remainder modulo p of the coefficient in front of x^n in the product PQ is 1 for $n = 0$ and $n = 4$; $p - 2$ for $n = 2$ and is 0 for all other $n \geq 0$?

---

5 The non-decreasing functions $f, g : \mathbb{R} \to \mathbb{R}$ are such that $f(r) \leq g(r)$ for $\forall$ rational numbers r. Is it true that $f(x) \leq g(x)$ for $\forall$ real numbers x?

---

6 Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that:
$xf(y) + yf(x) = (x + y)f(x^2 + y^2), \forall x, y \in \mathbb{N}$

---

7 Let n be a natural number. The graph G has $10n$ vertices. They are separated into 10 groups with n vertices and we know that there is an edge between two of them if and only if they belong to two different groups. Whats the greatest number of edges a subgraph of G can have, so that there are no 4-cliques in it?

---

8 We are given a ΔABC. Point D on the circumscribed circle k is such that CD is a symmedian in ΔABC. Let X and Y be on the rays $\overrightarrow{CB}$ and $\overrightarrow{CA}$, so that $CX = 2CA$ and $CY = 2CB$. Prove that the circle, tangent externally to k and to the lines CA and CB, is tangent to the circumscribed circle of ΔXDY.

– Final Round

---

1 Find the least value of $k \in \mathbb{N}$ with the following property: There doesnt exist an arithmetic progression with 2019 members, from which exactly k are integers.

---

2 Does there exist a strictly increasing function $f : \mathbb{N} \to \mathbb{N}$, such that for $\forall\, n \in \mathbb{N}$: $f(f(f(n))) = n + 2f(n)$?

---

3 ΔABC is isosceles with a circumscribed circle $\omega(O)$. Let H be the foot of the altitude from C to AB and let M be the middle point of AB. We define a point X as the second intersection point of the circle with diameter CM and ω and let XH intersect ω for a second time in Y. If $CO \cap AB = D$, then prove that the circumscribed circle of ΔYHD is tangent to ω.

---

4 The inscribed circle of an acute ΔABC is tangent to AB and AC in K and L respectively. The altitude AH intersects the angle bisectors of $\angle ABC$ and $\angle ACB$ in P and Q respectively. Prove that the middle point M of AH lies on the radical axis of the circumscribed circles of ΔKPB and ΔLQC.

---

5 For $\forall\ m \in \mathbb{N}$ with $\pi(m)$ we denote the number of prime numbers that are no bigger than m. Find all pairs of natural numbers (a, b) for which there exist polynomials $P, Q \in \mathbb{Z}[x]$ so that for $\forall\ n \in \mathbb{N}$ the following equation is true: $\frac{\pi(an)}{\pi(bn)} = \frac{P(n)}{Q(n)}$.

---

6 Prove that for $\forall\ z \in \mathbb{C}$ the following inequality is true: $|z|^2 + 2|z - 1| \geq 1$, where $" = "$ is reached when $z = 1$.

---

7 A convex polyhedron has m triangular faces (there can be faces of other kind too). From each vertex there are exactly 4 edges. Find the least possible value of m.

---

8 On an exam there are 5 questions, each with 4 possible answers. 2000 students went on the exam and each of them chose one answer to each of the questions. Find the least possible value of n, for which it is possible for the answers that the students gave to have the following property: From every n students there are 4, among each, every 2 of them have no more than 3 identical answers.
