International Mathematical Excellence Olympiad 2019

www.artofproblemsolving.com/community/c986225

by MK4J

1 Let ABC be a scalene triangle with circumcircle ω. The tangent to ω at A meets BC at D. The A-median of triangle ABC intersects BC and ω at M and N, respectively. Suppose that K is a point such that $ADMK$ is a parallelogram. Prove that $KA = KN$.

Proposed by Alexandru Lopotenco (Moldova)

---

2 Consider some graph G with 2019 nodes. Let's define *inverting* a vertex v the following process: for every other vertex u, if there was an edge between v and u, it is deleted, and if there wasn't, it is added. We want to minimize the number of edges in the graph by several *invertings* (we are allowed to invert the same vertex several times). Find the smallest number M such that we can always make the number of edges in the graph not larger than M, for any initial choice of G.

Proposed by Arsenii Nikolaev, Anton Trygub (Ukraine)

---

3 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real x, y, the following relation holds:

$$(x + y) \cdot f(x + y) = f(f(x) + y) \cdot f(x + f(y)).$$

Proposed by Vadym Koval (Ukraine)

---

4 Call a two-element subset of $\mathbb{N}$ *cute* if it contains exactly one prime number and one composite number. Determine all polynomials $f \in \mathbb{Z}[x]$ such that for every *cute* subset $\{p, q\}$, the subset $\{f(p) + q, f(q) + p\}$ is *cute* as well.

Proposed by Valentio Iverson (Indonesia)

---

5 Find all pairs of positive integers (s, t), so that for any two different positive integers a and b there exists some positive integer n, for which

$$a^s + b^t | a^n + b^{n+1}.$$

Proposed by Oleksii Masalitin (Ukraine)

---

6 Let ABC be a scalene triangle with incenter I and circumcircle ω. The internal and external bisectors of angle $\angle BAC$ intersect BC at D and E, respectively. Let M be the point on segment AC such that $MC = MB$. The tangent to ω at B meets MD at S. The circumcircles of triangles

ADE and BIC intersect each other at P and Q. If AS meets ω at a point K other than A, prove that K lies on PQ.

Proposed by Alexandru Lopotenco (Moldova)