National Math Olympiad (3rd Round) 2019

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– Mid-Terms

– Algebra

1 a,b and c are positive real numbers so that $\sum_{\text{cyc}}(a+b)^2=2\sum_{\text{cyc}}a+6abc$. Prove that

$$\sum_{\text{cyc}}(a-b)^2\leq\left|2\sum_{\text{cyc}}a-6abc\right|.$$

2 Find all function $f:\mathbb{R}\to\mathbb{R}$ such that for any three real number a,b,c , if $a+f(b)+f(f(c))=0$:

$$f(a)^3+bf(b)^2+c^2f(c)=3abc$$

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Proposed by Amirhossein Zolfaghari

3 We are given a natural number d. Find all open intervals of maximum length $I\subseteq R$ such that for all real numbers $a_0,a_1,...,a_{2d-1}$ inside interval I, we have that the polynomial $P(x)=x^{2d}+a_{2d-1}x^{2d-1}+...+a_1x+a_0$ has no real roots.

– Combinatorics

1 Hossna is playing with a $m*n$ grid of points.In each turn she draws segments between points with the following conditions.

1. No two segments intersect.

2. Each segment is drawn between two consecutive rows.

3. There is at most one segment between any two points.

Find the maximum number of regions Hossna can create.

2 Let n,k be positive integers so that $n\geq k$.Find the maximum number of binary sequances of length n so that fixing any arbitary k bits they do not produce all binary sequances of length k.For exmple if $k=1$ we can only have one sequance otherwise they will differ in at least one bit which means that bit produces all binary sequances of length 1.

3 Cells of a $n*n$ square are filled with positive integers in the way that in the intersection of the $i-$th column and $j-$th row, the number $i+j$ is written. In every step, we can choose two non-intersecting equal rectangles with one dimension equal to n and swap all the numbers inside these two rectangles with one another. (without reflection or rotation) Find the minimum number of moves one should do to reach the position where the intersection of the $i-$th column and $j-$row is written $2n+2-i-j$.

– Geometry

1 Given a cyclic quadrilateral $ABCD$. There is a point P on side BC such that $\angle PAB = \angle PDC = 90^\circ$. The medians of vertexes A and D in triangles PAB and PDC meet at K and the bisectors of $\angle PAB$ and $\angle PDC$ meet at L. Prove that $KL \perp BC$.

2 Consider an acute-angled triangle ABC with $AB = AC$ and $\angle A > 60^\circ$. Let O be the circumcenter of ABC. Point P lies on circumcircle of BOC such that $BP \parallel AC$ and point K lies on segment AP such that $BK = BC$. Prove that CK bisects the arc BC of circumcircle of BOC.

3 Consider a triangle ABC with circumcenter O and incenter I. Incircle touches sides BC, CA and AB at D, E and F. K is a point such that KF is tangent to circumcircle of BFD and KE is tangent to circumcircle of CED. Prove that BC, OI and AK are concurrent.

– Number Theory

1 Given a number $k \in \mathbb{N}$. $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ are two sequences of positive integers that $a_i, b_i \in \{1, 2, \cdots, 9\}$. For all $n \geq 0$

$$\overline{a_n \cdots a_1 a_0} + k \mid \overline{b_n \cdots b_1 b_0} + k \ .$$

Prove that there is a number $1 \leq t \leq 9$ and $N \in \mathbb{N}$ such that $b_n = t a_n$ for all $n \geq N$.

(Note that $(\overline{x_n x_{n-1} \ldots x_0}) = 10^n \times x_n + \cdots + 10 \times x_1 + x_0$)

2 Prove that for any positive integers $m > n$, there is infinitely many positive integers a, b such that set of prime divisors of $a^m + b^n$ is equal to set of prime divisors of $a^{2019} + b^{1398}$.

3 Let S be an infinite set of positive integers and define:

$T = \{x + y | x, y \in S, x \neq y\}$

Suppose that there are only finite primes p so that:

1.$p \equiv 1 \pmod 4$

2.There exists a positive integer s so that $p|s, s \in T$.

Prove that there are infinity many primes that divide at least one term of S.

–	Finals	
–	Algebra	

1 Let $A_1, A_2, \dots A_k$ be points on the unit circle.Prove that:

$$\sum_{1\le i<j\le k} d(A_i, A_j)^2 \le k^2$$

Where $d(A_i, A_j)$ denotes the distance between A_i, A_j.

2 $P(x)$ is a monoic polynomial with integer coefficients so that there exists monoic integer coefficients polynomials $p_1(x), p_2(x), \dots, p_n(x)$ so that for any natural number x there exist an index j and a natural number y so that $p_j(y) = P(x)$ and also $deg(p_j) \ge deg(P)$ for all j.Show that there exist an index i and an integer k so that $P(x) = p_i(x+k)$.

3 Let a, b, c be non-zero distinct real numbers so that there exist functions $f, g : \mathbb{R}^+ \to \mathbb{R}$ so that:

$$af(xy) + bf(\frac{x}{y}) = cf(x) + g(y)$$

For all positive real x and large enough y.

Prove that there exists a function $h : \mathbb{R}^+ \to \mathbb{R}$ so that:

$$f(xy) + f(\frac{x}{y}) = 2f(x) + h(y)$$

For all positive real x and large enough y.

– Combinatorics

1 A bear is in the center of the left down corner of a $100 * 100$ square .we call a cycle in this grid a bear cycle if it visits each square exactly ones and gets back to the place it started.Removing a row or column with compose the bear cycle into number of pathes.Find the minimum k so that in any bear cycle we can remove a row or column so that the maximum length of the remaining pathes is at most k.

2 Let T be a triangulation of a 100-gon.We construct $P(T)$ by copying the same 100-gon and drawing a diagonal if it was not drawn in T an there is a quadrilateral with this diagonal and two other vertices so that all the sides and diagonals(Except the one we are going to draw) are present in T.Let $f(T)$ be the number of intersections of diagonals in $P(T)$.Find the minimum and maximum of $f(T)$.

– Geometry

1 Consider a triangle ABC with incenter I. Let D be the intersection of BI, AC and CI intersects the circumcircle of ABC at M. Point K lies on the line MD and $\angle KIA = 90^\circ$. Let F be the reflection of B about C. Prove that $BIKF$ is cyclic.

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2 In acute-angled triangle ABC altitudes BE, CF meet at H. A perpendicular line is drawn from H to EF and intersects the arc BC of circumcircle of ABC (that doesn't contain A) at K. If AK, BC meet at P, prove that $PK = PH$.

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3 Given an inscribed pentagon $ABCDE$ with circumcircle Γ. Line ℓ passes through vertex A and is tangent to Γ. Points X, Y lie on ℓ so that A lies between X and Y. Circumcircle of triangle XED intersects segment AD at Q and circumcircle of triangle YBC intersects segment AC at P. Lines XE, YB intersects each other at S and lines XQ, YP at Z. Prove that circumcircle of triangles XYZ and BES are tangent.

– Number Theory

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1 Find all functions $f : \mathbb{N} \to \mathbb{N}$ so that for any distinct positive integers x, y, z the value of $x+y+z$ is a perfect square if and only if $f(x) + f(y) + f(z)$ is a perfect square.

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2 Call a polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \dots a_1 x + a_0$ with integer coefficients primitive if and only if $\gcd(a_n, a_{n-1}, \dots a_1, a_0) = 1$.

a)Let $P(x)$ be a primitive polynomial with degree less than 1398 and S be a set of primes greater than 1398.Prove that there is a positive integer n so that $P(n)$ is not divisible by any prime in S.

b)Prove that there exist a primitive polynomial $P(x)$ with degree less than 1398 so that for any set S of primes less than 1398 the polynomial $P(x)$ is always divisible by product of elements of S.

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3 Let a, m be positive integers such that $Ord_m(a)$ is odd and for any integers x, y so that

1.$xy \equiv a \pmod{m}$

2.$Ord_m(x) \leq Ord_m(a)$

3.$Ord_m(y) \leq Ord_m(a)$

We have either $Ord_m(x) | Ord_m(a)$ or $Ord_m(y) | Ord_m(a)$.prove that $Ord_m(a)$ contains at most one prime factor.
