Danube Mathematical Competition 2019

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–	Juniors
1	Solve in $\mathbb{Z}^2$ the equation: $x^2\left(1+x^2\right)=-1+21^y$. *Lucian Petrescu*
2	Let be a natural number n, and n real numbers $a_1, a_2, \dots, a_n$. Prove that there exists a real number a such that $a+a_1, a+a_2, \dots, a+a_n$ are all irrational.
3	Let be a sequence of 51 natural numbers whose sum is 100. Show that for any natural number $1 \le k < 100$ there are some consecutive numbers from this sequence whose sum is k or $100-k$.
4	Let $ABCD$ be a cyclic quadrilateral,M midpoint of AC and N midpoint of BD. If $\angle AMB = \angle AMD$, prove that $\angle ANB = \angle BNC$.
–	Seniors
1	Find all prime p numbers such that p^3-4p+9 is perfect square.
2	Find all nondecreasing functions $f:\mathbb{R}\longrightarrow\mathbb{R}$ that verify the relation $$f\left(f\left(x^2\right)+y+f(y)\right)=x^2+2f(y),$$ for any real numbers x, y.
3	We color some unit squares in a 99×99 square grid with one of 5 given distinct colors, such that each color appears the same number of times. On each row and on each column there are no differently colored unit squares. Find the maximum possible number of colored unit squares.
4	Let APD be an acute-angled triangle and let B, C be two points on the segments (excluding their endpoints) AP, PD, respectively. The diagonals of $ABCD$ meet at Q. Denote by H_1, H_2 the orthocenters of APD, BPC, respectively. The circumcircles of ABQ and CDQ intersect at $X \neq Q$, and the circumcircles of ADQ, BCQ meet at $Y \neq Q$. Prove that if the line H_1H_2 passes through X, then it also passes through Y.