British Mathematical Olympiad 2006

– November 30th

1 Let n be an integer greater than 6. Prove that if $n-1$ and $n+1$ are both prime, then $n^2(n^2+16)$ is divisible by 720. Is the converse true?

2 Adrian teaches a class of six pairs of twins. He wishes to set up teams for a quiz, but wants to avoid putting any pair of twins into the same team. Subject to this condition:
i) In how many ways can he split them into two teams of six?
ii) In how many ways can he split them into three teams of four?

3 In the cyclic quadrilateral $ABCD$, the diagonal AC bisects the angle DAB. The side AD is extended beyond D to a point E. Show that $CE = CA$ if and only if $DE = AB$.

4 The equilateral triangle ABC has sides of integer length N. The triangle is completely divided (by drawing lines parallel to the sides of the triangle) into equilateral triangular cells of side length 1. A continuous route is chosen, starting inside the cell with vertex A and always crossing from one cell to another through an edge shared by the two cells. No cell is visited more than once. Find, with proof, the greatest number of cells which can be visited.

5 Let G be a convex quadrilateral. Show that there is a point X in the plane of G with the property that every straight line through X divides G into two regions of equal area if and only if G is a parallelogram.

6 Let T be a set of 2005 coplanar points with no three collinear. Show that, for any point of the 2005 points, the number of triangles it lies strictly within, whose vertices are points in T, is even.